MTH 165: Linear Algebra with Differential Equations

1st Midterm

February 23, 2012

NAME (please print legibly): Solutions

Your University ID Number: ____________________

Indicate your instructor with a check in the box:

Dan-Andrei Geba	MWF 10:00 - 10:50 AM	
Ang Wei	MW 2:00 - 3:15 PM	

- **The presence of of electronic devices (including calculators), books, or formula cards/sheets at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Clearly circle or label your simplified final answers.**
- **You are responsible for checking that this exam has all 7 pages.**

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
TOTAL	60	

1. (10 points) Find the general solution for the equation

$$\frac{dy}{dt} + \frac{2t+1}{t}y = 2t.$$

May assume that solving for $t>0$.

Integrating factor

$$I(t) = e^{\int \frac{2t+1}{t}dt} = e^{\int (2+\frac{1}{t})dt} = e^{2t+\ln|t|} = te^{2t}$$

Multiply equation by $I(t)$ to get

$$te^{2t}\frac{dy}{dt} + e^{2t}(2t+1)y = 2t^2e^{2t}$$

$$\frac{d}{dt}\left(te^{2t}y\right) = 2t^2e^{2t}$$

integration by parts

$$te^{2t}y = \int 2t^2e^{2t}dt = t^2e^{2t} - \int 2te^{2t}dt$$

$$= t^2e^{2t} - \left(te^{2t} - \int e^{2t}dt\right)$$

$$= t^2e^{2t} - te^{2t} + \frac{1}{2}e^{2t} + C$$

Thus the general solution is

$$y(t) = t - 1 + \frac{1}{2t} + \frac{C}{te^{2t}}$$

2. (10 points) Solve the initial value problem

$$\frac{dy}{dx} = 2xy^2 + 3x^2y^2, \quad y(1) = -1.$$

$$\frac{dy}{dx} = (2x+3x^2)y^2$$

$$\int \frac{1}{y^2}\,dy = \int (2x+3x^2)\,dx$$

$$-\frac{1}{y} = x^2 + x^3 + C$$

$$\text{or} \quad y(x) = -\frac{1}{x^3+x^2+C}$$

Now use $y(1)=-1$ to get

$$-1 = -\frac{1}{1^3+1^2+C}$$

So $\quad C+2=1 \quad$ or $\quad C=-1$

Thus $\quad y(x) = -\dfrac{1}{x^3+x^2-1}$

Note $y(x)=0$ is an equilibrium solution to the equation but does not fulfill the initial condition. This justifies that we can divide by y.

3. (10 points) A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

We are given $V(0) = 100$ gal, $A(0) = 50$ lb, $c_1 = 1 \frac{\text{lb}}{\text{gal}}$, $r_1 = 5$ gal/s, $r_2 = 3$ gal/s.

$\frac{dV}{dt} = r_1 - r_2$ implies $\frac{dV}{dt} = 2$ so $V(t) = 2t + C$

Using $V(0) = 100$ we get $100 = 2 \cdot 0 + C$ so $C = 100$. Thus $V(t) = 2t + 100$. The tank is full when $400 = V(t)$ so $400 = 2t + 100$ or $t = 150$ s

Need to find $A(150)$. The equation for $A(t)$ is

$$\frac{dA}{dt} + \frac{r_2}{(r_1 - r_2)t + V(0)} A = c_1 r_1$$

or
$$\frac{dA}{dt} = \frac{3}{2t + 100} A = 5$$

Integrating factor is $e^{\int \frac{3}{2t+100} dt} = e^{\frac{3}{2} \int \frac{dt}{t+50}} = e^{\frac{3}{2} \ln|t+50|} = (t+50)^{3/2}$

Multiply the equation by it and get

$\frac{d}{dt}\left((t+50)^{3/2} A \right) = 5(t+50)^{3/2}$ which implies

$(t+50)^{3/2} A = \int 5(t+50)^{3/2} dt = 2(t+50)^{5/2} + C$ and thus

$A(t) = 2(t+50) + \frac{C}{(t+50)^{3/2}}$. Now $50 = A(0) = 2 \cdot (0+50) + \frac{C}{(0+50)^{3/2}}$ and thus $C = -50^{5/2}$. Thus $A(t) = 2t + 100 - \frac{50^{5/2}}{(t+50)^{3/2}}$. The amount of salt will thus be $A(150) = 400 - \frac{50^{5/2}}{200^{3/2}} = \frac{1575}{4}$ lb

4. (10 points) Find the rank for the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{bmatrix}$$

by computing its **reduced row-echelon form.**

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{bmatrix}$$

$$\overset{A_{12}(-2),\ A_{13}(2),\ A_{14}(-3)}{\sim} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -5 & -3 \end{bmatrix}$$

$$\overset{A_{21}(-2),\ A_{23}(-1),\ A_{24}(2)}{\sim} \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\overset{M_3(-1)}{\sim} \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\overset{A_{31}(5),\ A_{32}(-3),\ A_{34}(-1)}{\sim} \begin{bmatrix} 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{A_{41}(12),\ A_{42}(-7),\ A_{43}(2)}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus rank$(A) = 4$.

5. **(10 points)** Solve the following linear system of equations:

$$\begin{cases} x + y - z = 5 \\ 3x + y + 3z = 11 \\ 4x + y + 5z = 14 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 3 & 1 & 3 & 11 \\ 4 & 1 & 5 & 14 \end{array}\right] \underset{A_{13}(-4)}{\overset{A_{12}(-3)}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & -2 & 6 & -4 \\ 0 & -3 & 9 & -6 \end{array}\right]$$

$$\overset{M_2(-\frac{1}{2})}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -3 & 9 & -6 \end{array}\right]$$

$$\overset{A_{23}(3)}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Equivalent system $\begin{cases} x + y - z = 5 \\ y - 3z = 2 \end{cases}$

Set $z = t$. Then $y = 3t + 2$ and $x = 3 - 2t$

Solution set $\{(3-2t, 3t+2, t) : t \in \mathbb{R}\}$

or can write the solution as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3-2t \\ 3t+2 \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

6. (10 points) Find the inverse of the matrix

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{M_1(\tfrac{1}{4})} \left[\begin{array}{ccc|ccc} 1 & 3/4 & 1/2 & 1/4 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{A_{13}(-3)}{\overset{A_{12}(-5)}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 3/4 & 1/2 & 1/4 & 0 & 0 \\ 0 & 9/4 & 1/2 & -5/4 & 1 & 0 \\ 0 & 11/4 & 1/2 & -3/4 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{A_{23}(-1)} \left[\begin{array}{ccc|ccc} 1 & 3/4 & 1/2 & 1/4 & 0 & 0 \\ 0 & 9/4 & 1/2 & -5/4 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 & -1 & 1 \end{array}\right]$$

$$\underset{\sim}{P_{23}} \left[\begin{array}{ccc|ccc} 1 & 3/4 & 1/2 & 1/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & -1 & 1 \\ 0 & 9/4 & 1/2 & -5/4 & 1 & 0 \end{array}\right]$$

$$\underset{\sim}{M_2(2)} \left[\begin{array}{ccc|ccc} 1 & 3/4 & 1/2 & 1/4 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 9/4 & 1/2 & -5/4 & 1 & 0 \end{array}\right]$$

$$\underset{A_{23}(-\frac{9}{4})}{\overset{A_{21}(-\frac{3}{4})}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & -1/2 & 3/2 & -3/2 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1/2 & -\frac{14}{4} & \frac{11}{2} & -\frac{9}{2} \end{array}\right]$$

$$\underset{\sim}{M_3(2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & -1/2 & 3/2 & -3/2 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -\underset{7}{\cancel{14/2}} & 11 & -9 \end{array}\right]$$

$$\underset{\sim}{A_{31}(-\tfrac{1}{2})} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 3 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -\underset{7}{\cancel{14/2}} & 11 & -9 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{bmatrix}$$