MTH 165: Linear Algebra with Differential Equations

2nd Midterm

April 5, 2012

NAME (please print legibly): Solutions

Your University ID Number: ______________________

Indicate your instructor with a check in the box:

Dan-Andrei Geba	MWF 10:00 - 10:50 AM	
Ang Wei	MW 2:00 - 3:15 PM	

- **The presence of of electronic devices (including calculators), books, or formula cards/sheets at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Clearly circle or label your simplified final answers.**
- **You are responsible for checking that this exam has all 7 pages.**

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
TOTAL	60	

1. (10 points) Find the determinants of the matrices A, B, and $B^T A$, where

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 2 & 2 & 1 \\ -2 & 0 & 4 & 1 \\ 0 & -2 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 5 & 6 & -14 \\ 0 & 2 & 13 & -156 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

$$\det(A) = 1 \cdot \begin{vmatrix} 2 & 2 & 1 \\ 0 & 4 & 1 \\ -2 & 3 & 4 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & -1 & 1 \\ 0 & 4 & 1 \\ -2 & 3 & 4 \end{vmatrix} + (-2) \begin{vmatrix} -1 & -1 & 1 \\ 2 & 2 & 1 \\ -2 & 3 & 4 \end{vmatrix} - 0 \cdot \begin{vmatrix} -1 & -1 & 1 \\ 2 & 2 & 1 \\ 0 & 4 & 1 \end{vmatrix}$$

$$= \left(2 \begin{vmatrix} 4 & 1 \\ 3 & 4 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}\right)$$

$$- \left((-1) \begin{vmatrix} 4 & 1 \\ 3 & 4 \end{vmatrix} - 0 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} + (-2) \begin{vmatrix} -1 & 1 \\ 4 & 1 \end{vmatrix}\right)$$

$$- 2 \left((-1) \cdot \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 2 & 1 \\ -2 & 4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix}\right)$$

$$= 2 \cdot (16-3) - 2(2-4) - \left(-(16-3) - 2 \cdot (-1-4)\right)$$

$$-2\left(-(8-3) + (8+2) + (6+4)\right)$$

$$= 26 + 4 + 13 - 10 + 10 - 20 - 20$$

$$= 3$$

$$\det(B) = (-3) \cdot 2 \cdot \left(-\tfrac{1}{3}\right) \cdot 5 = 10$$

$$\det(B^T A) = \det(B^T) \det(A) = \det(B) \det(A)$$

$$= 10 \cdot 3 = 30$$

2. (10 points) In each of the following, determine whether the subset S is a subspace of the given vector space V:

i) $V = M_{2\times 2}(\mathbb{R})$ and S is the subset of all 2×2 invertible matrices;

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible and is thus not in S.

S is thus not a subspace.

ii) $V = P_2$, the vector space of real-valued polynomials of degree ≤ 2, and

$$S = \{ax^2 + bx \,:\, a, b \in \mathbb{R}\}.$$

$0 = 0 \cdot x^2 + 0 \cdot x \in S$ so $S \neq \emptyset$

Consider $a_1x^2 + b_1x$, $a_2x^2 + b_2x$, $k \in \mathbb{R}$

Then

$$(a_1x^2 + b_1x) + (a_2x^2 + b_2x) = (a_1 + a_2)x^2 + (b_1 + b_2)x \in S$$

and

$$k(a_1x^2 + b_1x) = (ka_1)x^2 + (kb_1)x \in S$$

Thus closed under addition and scalar multiplication so is a subspace.

3. (10 points) Compute

$$\operatorname{span}\{(1,0,-1),(2,0,4),(-5,0,2),(0,0,1)\}$$

in the vector space $\mathbb{R}^3$.

$$\begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 4 & 2 & 1 \end{bmatrix}$$

$$\overset{P_{23}}{\sim} \begin{bmatrix} 1 & 2 & -5 & 0 \\ -1 & 4 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{A_{12}(1)}{\sim} \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 6 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{M_2(\frac{1}{6})}{\sim} \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -1/2 & 1/6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \ \uparrow$

leading 1's

Thus

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \right\}$$

4. (10 points) Using the Wronskian, determine whether or not the functions

$$f_1(x) = e^{2x}, \; f_2(x) = e^{3x}, \; f_3(x) = e^{-x}$$

are linearly independent on $\mathbb{R}$.

$$W[f_1, f_2, f_3](x) = \begin{vmatrix} e^{2x} & e^{3x} & e^{-x} \\ 2e^{2x} & 3e^{3x} & -e^{-x} \\ 4e^{2x} & 9e^{3x} & e^{-x} \end{vmatrix}$$

so $$W[f_1, f_2, f_3](0) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 4 & 9 & 1 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 3 & -1 \\ 9 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix}$$

$$= (3+9) - (2+4) + (18-12)$$

$$= 12 - 6 + 6 = 12 \neq 0$$

so the functions are linearly independent on $\mathbb{R}$

5. (10 points) Let S be the subspace of $\mathbb{R}^3$ that consists of all (x, y, z) which satisfy the equation $x + 3y - 2z = 0$. Determine a basis for S and find $\dim[S]$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3y+2z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 2z \\ 0 \\ z \end{bmatrix} = y\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ span S

They are also linearly independent because

$c_1 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 0$ implies $c_1 = 0, c_2 = 0$

using rows 2 and 3.

Thus $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ basis for S

and hence $\dim(S) = 2$.

6. (10 points) For the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & 5 \\ 0 & 2 & -1 & 7 \\ 4 & 2 & -3 & 13 \end{bmatrix},$$

find:

i) a basis and the dimension for colspace (A);

$$\begin{bmatrix} 1 & 1 & -1 & 5 \\ 0 & 2 & -1 & 7 \\ 4 & 2 & -3 & 13 \end{bmatrix} \overset{A_{13}(-4)}{\sim} \begin{bmatrix} 1 & 1 & -1 & 5 \\ 0 & 2 & -1 & 7 \\ 0 & -2 & 1 & -7 \end{bmatrix} \overset{A_{23}(-1)}{\sim} \begin{bmatrix} 1 & 1 & -1 & 5 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{M_2(\frac{1}{2})}{\sim} \begin{bmatrix} 1 & 1 & -1 & 5 \\ 0 & 1 & -\frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

leading 1's

Thus $\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$ basis

and dimension is 2

ii) a basis and the dimension for nullspace (A).

Need to solve $A\vec{x} = \vec{0}$ $\quad \left[\begin{array}{cccc|c} 1 & 1 & -1 & 5 & 0 \\ 0 & 2 & -1 & 7 & 0 \\ 4 & 2 & -3 & 13 & 0 \end{array}\right]$

do same row operations as above, get $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 5 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Set $x_3 = s$, $x_4 = t$ free variables. Then

$x_2 = \frac{1}{2}s - \frac{7}{2}t$ and

$x_1 = -x_2 + x_3 - 5x_4 = -\left(\frac{1}{2}s - \frac{7}{2}t\right) + s - 5t = \frac{1}{2}s + \frac{7}{2}t$

Thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7/2 \\ -7/2 \\ 0 \\ 1 \end{bmatrix}$. Thus $\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7/2 \\ -7/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

spans the nullspace but also linearly independent as seen from rows 3 and 4 and is thus a basis. dimension is 2