MTH 165: Linear Algebra with Differential Equations

Final Exam

May 6, 2013

NAME (please print legibly): Solutions

Your University ID Number: ______________________

Indicate your instructor with a check in the box:

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- **The presence of electronic devices (including calculators), books, or formula cards/sheets at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Clearly circle or label your simplified final answers.**
- **You are responsible for checking that this exam has all 11 pages.**

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

1. (10 points) Solve the following initial value problems:

(a) (5 points)

$$(x^2+1)\frac{dy}{dx} + 3x\,y = 6x, \quad y(0) = 3;$$

$(x^2+1)\dfrac{dy}{dx} = 6x - 3xy = 3x(2-y)$. So $\dfrac{dy}{2-y} = \dfrac{3x}{x^2+1}$

Integrating $\int \dfrac{dy}{2-y} = \int \dfrac{3x}{x^2+1} \Rightarrow -\ln|2-y| = \frac{3}{2}\ln(x^2+1) + C$

Set $x=0$: $-\ln|2-3| = \frac{3}{2}\ln(1) + C \Rightarrow C = 0.$

So $\ln|2-y| = -\frac{3}{2}\ln(x^2+1) = \ln\left[(x^2+1)^{-3/2}\right]$

$|2-y| = (x^2+1)^{-3/2}$. Looking at $x=0$ where $y=3$ and not $y=1$ $\rightsquigarrow$ $y-2 = (x^2+1)^{-3/2}$ or

$$\boxed{y = 2 + (x^2+1)^{-3/2}}$$

(b) (5 points)

$$y' + \frac{y}{x^2} = \frac{2}{x^2}, \quad y(1) = 1.$$

Integrating factor $I(x) = e^{\int \frac{dx}{x^2}} = e^{-1/x}$.

Multiply both sides of the equation:

$e^{-1/x}y' + \dfrac{e^{-1/x}}{x^2}y = 2\dfrac{e^{-1/x}}{x^2} \Rightarrow \left[e^{-1/x}y\right]' = 2\dfrac{e^{-1/x}}{x^2}$

Integrating: $e^{-1/x}y = 2\int \dfrac{e^{-1/x}}{x^2}\,dx \overset{u=-1/x}{=} 2e^{-1/x} + C$

Setting $x=1$: $e^{-1}\cdot 1 = 2\cdot e^{-1} + C \Rightarrow C = -e^{-1}$.

So $e^{-1/x}y = 2e^{-1/x} - e^{-1} \Rightarrow \boxed{y = 2 - e^{-1}e^{1/x}}$

Note: Can also separate variables!

2. (10 points)

(a) (5 points) Find the value of k which satisfies

$$\det \begin{bmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{bmatrix} = k \cdot \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

$$\begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{vmatrix} = 2 \cdot 7 \cdot \begin{vmatrix} a_1 & a_2 & a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= 2 \cdot 7 \begin{vmatrix} a_1 & a_2 & a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= 2 \cdot 7 \cdot 3 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

So $\boxed{k = 42}$

(b) (5 points) Construct two matrices A and B of appropriate dimensions such that

$$\operatorname{rank}(AB) < \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}.$$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\operatorname{rank}(A) = \operatorname{rank}(B) = 1$ so $\min\{\operatorname{rank}(A), \operatorname{rank}(B)\} = 1$

However $\operatorname{rank}(AB) = 0 < 1$.

3. (10 points) For each of the following subsets of P_2 (i.e., the vector space of polynomials with real coefficients and degree at most 2) determine whether it is a subspace. If that is the case, find its dimension.

(a) (5 points) S is the set of polynomials $p \in P_2$ satisfying

$$p'(x) + p(x) = x^2.$$

Not a subspace. The zero polynomial $O(x)$ does not lie in S:

$O'(x) + O(x) = 0 + 0 = 0$

(b) (5 points) S is the set of polynomials $p \in P_2$ verifying

$$p(x) + p(-x) = 0.$$

This is a subspace.

(i) $O(x) \in S$ because $O(x) + O(-x) = 0 + 0 = 0$ ✓

(ii) Let $p, q \in S$. Then $(p+q) \in S$ because

$(p+q)(x) + (p+q)(-x) = p(x) + q(x) + p(-x) + q(-x)$

$= (p(x) + p(-x)) + (q(x) + q(-x)) = 0 + 0 = 0.$

(iii) Let $p \in S$, $\lambda \in \mathbb{R}$. Then $(\lambda p) \in S$ because

$(\lambda p)(x) + (\lambda p)(-x) = \lambda p(x) + \lambda p(-x) = \lambda (p(x) + p(-x)) = \lambda \cdot 0 = 0.$

To find a basis we determine a minimal spanning set. Suppose $p(x) \in S$. Then

Take $p(x) = ax^2 + bx + c$.

$0 = p(x) + p(-x) = ax^2 + bx + c + a(-x)^2 + b(-x) + c$

$= 2ax^2 + 2c$

Therefore must have $a = c = 0$. So S is the span of $\{x\}$. So $\{x\}$ is a basis for S and $\dim(S) = 1$.

4. (10 points) Compute the reduced row-echelon form for the matrix

$$\begin{bmatrix} 7 & 4 & 1 & 7 \\ 4 & 3 & 2 & 4 \\ 3 & 2 & 1 & 3 \end{bmatrix}$$

and deduce from there a basis and the dimension of its row space.

$$\begin{bmatrix} 7 & 4 & 1 & 7 \\ 4 & 3 & 2 & 4 \\ 3 & 2 & 1 & 3 \end{bmatrix} \overset{R_1 = -r_1 + r_2 + r_3}{\sim} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 4 & 3 & 2 & 4 \\ 3 & 2 & 1 & 3 \end{bmatrix} \overset{R_2 = r_2 - r_3}{\sim} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 3 \end{bmatrix}$$

$$\overset{\substack{R_1 = r_2 \\ R_2 = r_1}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 3 & 2 & 1 & 3 \end{bmatrix} \overset{R_3 = r_3 - 3r_1}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$$\overset{R_3 = r_2 + r_3}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_1 = r_1 - r_2}{\sim} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Is the reduced row-echelon form.

Basis for row space = set of non-zero rows of the reduced row-echelon form $= \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$.

Dimension of row space $= 2$.

5. (10 points) Let $M_{2\times 2}(\mathbb{R})$ denote the vector space of 2×2 square matrices with real entries. Define $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ c & a+d \end{pmatrix}.$$

(a) (3 points) Show that T is a linear transformation.

$$T\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right] = T\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} a+e+b+f & 0 \\ c+g & a+e+d+h \end{pmatrix}$$

$$= \begin{pmatrix} a+b & 0 \\ c & a+d \end{pmatrix} + \begin{pmatrix} e+f & 0 \\ g & e+h \end{pmatrix} = T\begin{pmatrix} a & b \\ c & d \end{pmatrix} + T\begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

$$T\left[\lambda\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = T\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} = \begin{pmatrix} \lambda a+\lambda b & 0 \\ \lambda c & \lambda a+\lambda d \end{pmatrix} = \lambda\begin{pmatrix} a+b & 0 \\ c & a+d \end{pmatrix} = \lambda T\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(b) (3 points) Determine a basis for Ker(T). What is dim[Ker(T)]?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Ker}(T) \iff T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff \begin{pmatrix} a+b & 0 \\ c & a+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\iff -a = b = d,\ c = 0.$$

$$\iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

So $\boxed{\left\{\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\right\} \text{ is a basis}}$ for Ker(T). Its $\boxed{\text{dimension is } 1}$.

(c) (2 points) Find dim[Rng(T)].

By Rank-Nullity:

$$\dim M_{2\times 2}(\mathbb{R}) = \dim(\mathrm{Ker}\,T) + \dim(\mathrm{Rng}(T))$$

$$4 = 1 + \dim(\mathrm{Rng}(T)) \text{ so}$$

$$\boxed{\dim(\mathrm{Rng}(T)) = 3}$$

(d) (2 points) Is the matrix $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ in the range of T? Explain.

Yes it is. $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2-1 & 0 \\ 2 & 2+1 \end{pmatrix} = T\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}.$

or $T\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$ etc..

6. (10 points) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) (4 points) Determine its eigenvalues and their multiplicities.

Characteristic polynomial $p(t) = \det(A - tI) = \begin{vmatrix} -t & 1 & 1 \\ 1 & -t & 1 \\ 1 & 1 & -t \end{vmatrix}$

$= -t \begin{vmatrix} -t & 1 \\ 1 & -t \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -t \end{vmatrix} + 1 \begin{vmatrix} 1 & -t \\ 1 & 1 \end{vmatrix} = -t(t^2-1) - (-t-1) + (1+t)$

$= -t(t-1)(t+1) + 2(t+1) = (t+1)(-t(t-1)+2) = -(t+1)(t^2-t-2)$

$= -(t+1)^2(t-2)$ Eigenvalues: $\lambda = -1$ multiplicity 2, $\lambda = 2$ multiplicity 1

(b) (4 points) Compute the eigenspaces corresponding to each of the eigenvalues and their dimensions.

$\lambda = -1$: $(A+I)\vec{v} = \vec{0} \iff \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ x_2, x_3 free, $x_1 = -x_2 - x_3$

so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ so eigenspace $= \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ of dimension 2.

$\lambda = 2$: $(A - 2I)\vec{v} = \vec{0} \iff \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \overset{R_1 = R_1 + R_2 + R_3}{\sim} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$

$\overset{R_1 = -R_2 + R_3, \; R_3 = R_1}{\sim} \left[\begin{array}{ccc|c} 0 & 3 & -3 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ x_3 free, $x_2 = x_3$, $x_1 = x_3$

so eigenspace $= \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ of dimension 1.

(c) (2 points) Conclude, with explanation, whether A is a defective or non-defective matrix.

A is non-defective because the sum of the dimensions of its eigenspaces equals 3.
A is 3x3 matrix.

7. (10 points) Solve the initial value problem

$$y''' - 4y'' + 5y' - 2y = 0, \quad y(0) = 2, \quad y'(0) = 3, \quad y''(0) = 5.$$

Axiliary polynomial: $p(r) = r^3 - 4r^2 + 5r - 2$

$r = 1$ is a root. $p(r) = (r-1)\,(r^2 - 3r + 2)$ by long division. (*)

$= (r-1)\,(r-1)(r-2)$

Roots $r = 1$ multiplicity 2

$r = 2$ — — 1.

General solution is $y = c_1 e^t + c_2 e^t t + c_3 e^{2t}$

$y'(t) = c_1 e^t + c_2 (t+1) e^t + 2c_3 e^{2t}$

$y''(t) = c_1 e^t + c_2 (t+2) e^t + 4c_3 e^{2t}$

Setting $t = 0$:
$2 = y(0) = c_1 + 0 + c_3$

$3 = y'(0) = c_1 + c_2 + 2c_3$

$5 = y''(0) = c_1 + 2c_2 + 4c_3$

By inspection get $c_1 = c_3 = 1, \quad c_2 = 0.$

So $\boxed{y(t) = e^t + e^{2t}}$

(*) The long division

$$\begin{array}{l|l} r^3 - 4r^2 + 5r - 2 & r - 1 \\ \cline{2-2} r^3 - r^2 & r^2 - 3r + 2 \\ \hline -3r^2 + 5r - 2 & \\ -3r^2 + 3r & \\ \hline 2r - 2 & \\ 2r - 2 & \\ \hline 0. & \end{array}$$

8. (10 points) Determine the general solution to

$$y'' + 4y' + 4y = x\,e^{-x}.$$

$y = y_c + y_p$.

$\underline{y_c}$: Auxiliary polynomial is $p(r) = r^2 + 4r + 4$
$= (r+2)^2$.

so $y_c = c_1 e^{-2x} + c_2 x e^{-2x}$

$\underline{y_p}$: For the particular solution we try

$y_p = A_0 e^{-x} + A_1 x e^{-x}$.

$y_p' = -A_0 e^{-x} + A_1 (1-x) e^{-x}$

$y_p'' = A_0 e^{-x} + A_1 (x-2) e^{-x}$

Substituting in the equation,

$A_0 e^{-x} + A_1 (x-2) e^{-x} \cancel{-4A_0 e^{-x}} + 4A_1 (1-x) e^{-x} \cancel{+4A_0 e^{-x}} + 4A_1 x e^{-x}$
$=$
$x e^{-x}$. So

$(A_0 - 2A_1 + 4A_1) e^{-x} + (A_1 - 4A_1 + 4A_1) x e^{-x} = x e^{-x}$. $\Rightarrow$

$(A_0 + 2A_1) e^{-x} + A_1 x e^{-x} = x e^{-x}$.

Equating coefficients of e^{-x}: $A_0 + 2A_1 = 0$
$x e^{-x}$: $A_1 = 1$

$\Rightarrow$ $A_0 = -2$
$A_1 = 1$

So $\boxed{y(x) = c_1 e^{-2x} + c_2 x e^{-x} - 2e^{-x} + x e^{-x}}$

9. (10 points) Find the general solution to

$$y'' + 2y' = e^{-x} + x.$$

$y = y_c + y_p$

$\underline{y_c}$: Auxiliary polynomial $p(r) = r^2 + 2r = r(r+2)$

So $y_c = c_1 + c_2 e^{-2x}$.

$\underline{y_p}$: Try $y_p = Ae^{-x} + \underline{Bx + Cx^2}$ Zero is a root of $p(r)$.

$y_p' = -Ae^{-x} + B + 2Cx$

$y_p'' = Ae^{-x} + 2C$.

Substituting

$Ae^{-x} + 2c - 2Ae^{-x} + 2B + 4Cx = e^{-x} + x$. So

$-Ae^{-x} + 4Cx + 2C + 2B = e^{-x} + x$.

Equating coefficients of e^{-x}: $-A = 1 \Rightarrow \boxed{A = -1}$

x: $4C = 1 \Rightarrow \boxed{C = 1/4}$

$\underline{1}$: $2C + 2B = 0 \Rightarrow \boxed{B = -1/4}$

So $\boxed{y(x) = c_1 + c_2 e^{-x} - e^{-x} - \frac{x}{4} + \frac{x^2}{4}.}$

10. (10 points) Solve the initial value problem

$$\begin{cases} x_1' = -2x_1 + x_2, \quad x_2' = x_1 - 2x_2, \\ x_1(0) = 3, \quad x_2(0) = 1. \end{cases}$$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ & the system becomes

$\vec{x}\,' = A\vec{x}$, $\vec{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

We find the eigenvalues / eigenvectors of A.

~~Auxiliary~~ Characteristic polynomial

$$p(t) = \det(A - tI) = \begin{vmatrix} -2-t & 1 \\ 1 & -2-t \end{vmatrix} = (2+t)^2 - 1^2$$

$$= (2+t-1)(2+t+1) = (1+t)(3+t)$$

$$\boxed{\lambda = -1, -3.}$$

$\lambda = -1$: $(A+I)\vec{v} = \vec{0} \iff \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow x_1 = x_2 \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -3$: $(A+3I)\vec{v} = \vec{0} \iff \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \Rightarrow x_1 = -x_2 \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Setting $t = 0$ gives $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 2 \\ c_2 = 1 \end{matrix}$

So $\vec{x} = 2e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

In terms of x_1, x_2

$$\begin{cases} x_1 = 2e^{-t} + e^{-3t} \\ x_2 = 2e^{-t} - e^{-3t} \end{cases}$$