MTH 165: Linear Algebra with Differential Equations

1st Midterm

February 21, 2013

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Indicate your instructor with a check in the box:

Dan-Andrei Geba	MWF 10:00 - 10:50	
Giorgis Petridis	MWF 13:00 - 13:50	
Eyvindur Ari Palsson	MW 14:00 - 15:15	

- The presence of of electronic devices (including calculators), books, or formula cards/sheets at this exam is strictly forbidden.
- Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Clearly circle or label your simplified final answers.
- You are responsible for checking that this exam has all 7 pages.

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
TOTAL	60	

1. (10 points) Find the explicit solution for the initial value problem

$$\frac{dy}{dx} = x^2 + x^2y^2, \quad y(0) = 1.$$

$$\frac{dy}{dx} = x^2(1+y^2)$$

$$\frac{dy}{1+y^2} = x^2 dx$$

$$\arctan y = \frac{x^3}{3} + C$$

$$y(0)=1 \Rightarrow \frac{\pi}{4} = 0 + C \Rightarrow C = \frac{\pi}{4}$$

$$\arctan y = \frac{x^3}{3} + \frac{\pi}{4}$$

$$y = \tan\left(\frac{x^3}{3} + \frac{\pi}{4}\right)$$

2. (10 points) Solve the initial value problem

$$\frac{dy}{dt} + \frac{y}{2} - \frac{e^{t/3}}{2} = 0, \quad y(0) = \frac{6}{5}.$$

$$\frac{dy}{dt} + \frac{1}{2} y = \frac{e^{t/3}}{2} \qquad p(t) = \frac{1}{2} \qquad g(t) = \frac{e^{t/3}}{2}$$

$$\text{int. factor} = e^{\int p(t)\,dt} = e^{t/2}$$

$$\left(y \cdot e^{\int p}\right)' = g \cdot e^{\int p} \Leftrightarrow$$

$$\Leftrightarrow \left(y \cdot e^{t/2}\right)' = \frac{e^{5t/6}}{2} \Rightarrow$$

$$\Rightarrow y \cdot e^{t/2} = \int \frac{e^{5t/6}}{2} = \frac{3}{5} e^{5t/6} + C$$

$$y(0) = \frac{6}{5} \Rightarrow \frac{6}{5} \cdot 1 = \frac{3}{5} \cdot 1 + C \Rightarrow C = \frac{3}{5}$$

$$\Rightarrow y \cdot e^{t/2} = \frac{3}{5} e^{5t/6} + \frac{3}{5}$$

$$\Rightarrow y = \frac{3}{5} e^{t/3} + \frac{3}{5} e^{-t/2}$$

3. (10 points) Consider the RC circuit which has

$$R = 2\,\Omega, \qquad C = \frac{1}{8}\,F, \quad \text{and} \quad E(t) = 5\,V.$$

If $q(0) = 7$ coulombs, determine the current in the circuit for $t \geq 0$.

$$R \cdot i(t) + \frac{q(t)}{C} = E(t) \qquad i(t) = q'(t)$$

$$2q' + 8q = 5 \Rightarrow q' + 4q = \frac{5}{2}$$

$$\text{int. factor} = e^{\int 4dt} = e^{4t}$$

$$(q \cdot e^{4t})' = \frac{5}{2} \cdot e^{4t}$$

$$q \cdot e^{4t} = \frac{5}{8} e^{4t} + C$$

$$q(0) = 7 \Rightarrow 7 \cdot 1 = \frac{5}{8} \cdot 1 + C \Rightarrow C = \frac{51}{8}$$

$$q \cdot e^{4t} = \frac{5}{8} e^{4t} + \frac{51}{8} \Rightarrow$$

$$\Rightarrow q = \frac{5}{8} + \frac{51}{8} e^{-4t} \Rightarrow$$

$$\Rightarrow i = q' = -\frac{51}{2} e^{-4t}$$

4. (10 points) A 200-gal tank initially contains 100 gal of pure water. Brine enters the tank through two faucets: one containing 0.2 lb/gal of salt flows in at the rate of 1 gal/min, while the second one containing 0.1 lb/gal of salt flows in at the rate of 3 gal/min. The well-stirred mixture flows out of the tank at the rate of 2 gal/min. How much salt is in the tank just before the solution overflows?

$V(0) = 100$

$A(0) = 0$

$$\begin{cases} \frac{dA}{dt} = 0.2 \cdot 1 + 0.1 \cdot 3 - \frac{A(t)}{V(t)} \cdot 2 \\ \frac{dV}{dt} = 2 \end{cases}$$

$V(t) = 100 + 2t$

$$A' = 0.5 - \frac{A}{\cancel{100+2t}\ 50+t} \cancel{2} \Rightarrow \begin{cases} A' + \frac{1}{50+t} A = 0.5 \\ A(0) = 0 \end{cases}$$

$$\text{int factor} = e^{\int \frac{1}{50+t} dt} = 50+t$$

$$(A \cdot (50+t))' = 0.5(50+t) = 25 + 0.5t$$

$$\Rightarrow A \cdot (50+t) = 25t + 0.25t^2 + C$$

$$A(0) = 0 \Rightarrow 0 \cdot 50 = 25 \cdot 0 + 0.25 \cdot 0^2 + C \Rightarrow C = 0$$

$$\Rightarrow A(t) = \frac{25t + 0.25t^2}{50+t}$$

Tank overflows when $V(t) = 200 \Rightarrow t = 50$

Therefore $A(50) = 18.75$ lb.

5. (10 points) Find the rank for the matrix

$$A = \begin{bmatrix} 5 & 2 & -5 \\ 9 & 4 & -7 \\ 4 & 1 & -7 \end{bmatrix}$$

by computing its **reduced row-echelon form.**

$$\begin{bmatrix} 5 & 2 & -5 \\ 9 & 4 & -7 \\ 4 & 1 & -7 \end{bmatrix} \xrightarrow{R1 \to R1 - R3} \begin{bmatrix} 1 & 1 & 2 \\ 9 & 4 & -7 \\ 4 & 1 & -7 \end{bmatrix} \xrightarrow[R3 \to R3 - 4R1]{R2 \to R2 - 9R1}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -25 \\ 0 & -3 & -15 \end{bmatrix} \xrightarrow{R2 \to R2/-5} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 5 \\ 0 & -3 & -15 \end{bmatrix} \longrightarrow$$

$$\xrightarrow{R3 \to R3 + 3R2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 - R2} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

rank = 2

6. (10 points) Solve the following linear system of equations:

$$\begin{cases} 3x_1 + x_2 + x_3 + 6x_4 = 14 \\ x_1 - 2x_2 + 5x_3 - 5x_4 = -7 \\ 4x_1 + x_2 + 2x_3 + 7x_4 = 17 \end{cases}$$

$$\left[\begin{array}{cccc|c} 3 & 1 & 1 & 6 & 14 \\ 1 & -2 & 5 & -5 & -7 \\ 4 & 1 & 2 & 7 & 17 \end{array}\right] \xrightarrow{R1 \to R1 - 2R2} \left[\begin{array}{cccc|c} 1 & 5 & -9 & 16 & 28 \\ 1 & -2 & 5 & -5 & -7 \\ 4 & 1 & 2 & 7 & 17 \end{array}\right]$$

$$\xrightarrow[R3 \to R3 - 4R1]{R2 \to R2 - R1} \left[\begin{array}{cccc|c} 1 & 5 & -9 & 16 & 28 \\ 0 & -7 & 14 & -21 & -35 \\ 0 & -19 & +38 & -57 & -95 \end{array}\right] \xrightarrow{R2 \to R2/-7}$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 5 & -9 & 16 & 28 \\ 0 & 1 & -2 & 3 & 5 \\ 0 & -19 & 38 & -57 & -95 \end{array}\right] \xrightarrow{R3 \to R3 + 19 \cdot R2}$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 5 & -9 & 16 & 28 \\ 0 & 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R1 \to R1 - 5R2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_2 bound var.

x_3, x_4 free var.

$$\begin{cases} x_3 = \alpha \\ x_4 = \beta \end{cases} \Longrightarrow \begin{cases} x_1 = -\alpha - \beta + 3 \\ x_2 = 2\alpha - 3\beta + 5 \end{cases}$$