MTH 165: Linear Algebra with Differential Equations

2nd Midterm

April 4, 2013

NAME (please print legibly): Solutions

Your University ID Number: ______________________

Indicate your instructor with a check in the box:

Dan-Andrei Geba	MWF 10:00 - 10:50	
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- The presence of of electronic devices (including calculators), books, or formula cards/sheets at this exam is strictly forbidden.
- Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Clearly circle or label your simplified final answers.
- You are responsible for checking that this exam has all 7 pages.

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
TOTAL	60	

1. (10 points) Find the inverse of the matrix

$$A = \begin{bmatrix} -7 & -3 & 1 \\ 2 & 1 & 0 \\ -28 & -13 & 3 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} -7 & -3 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -28 & -13 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1 \to R1+4R2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 4 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -28 & -13 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3 \to R3+28R1]{R2 \to R2-2R1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 4 & 0 \\ 0 & -1 & -2 & -2 & -7 & 0 \\ 0 & 15 & 31 & 28 & 112 & 1 \end{array}\right] \xrightarrow{R_2 \to (-1)R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 4 & 0 \\ 0 & 1 & 2 & 2 & 7 & 0 \\ 0 & 15 & 31 & 28 & 112 & 1 \end{array}\right] \xrightarrow{R3 \to R3-15R2}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 4 & 0 \\ 0 & 1 & 2 & 2 & 7 & 0 \\ 0 & 0 & 1 & -2 & 7 & 1 \end{array}\right] \xrightarrow[R2 \to R2-2R3]{R1 \to R1-R3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -3 & -1 \\ 0 & 1 & 0 & 6 & -7 & -2 \\ 0 & 0 & 1 & -2 & 7 & 1 \end{array}\right] \xrightarrow{R_1 \to R1-R2}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 4 & 1 \\ 0 & 1 & 0 & 6 & -7 & -2 \\ 0 & 0 & 1 & -2 & 7 & 1 \end{array}\right]$$

Thus $A^{-1} = \begin{bmatrix} -3 & 4 & 1 \\ 6 & -7 & -2 \\ -2 & 7 & 1 \end{bmatrix}$

2. **(10 points)** Use cofactor expansion and/or row reduction to evaluate the determinant of the following matrix

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ -2 & 2 & -2 & 2 \\ 2 & 1 & -1 & -2 \\ -1 & -4 & 4 & 2 \end{bmatrix}.$$

$$\begin{vmatrix} 1 & 2 & 2 & 4 \\ -2 & 2 & -2 & 2 \\ 2 & 1 & -1 & -2 \\ -1 & -4 & 4 & 2 \end{vmatrix} \underset{\substack{R3 \to R3 - 2R1 \\ R4 \to R4 + R1}}{\overset{R2 \to R2 + 2R1}{=\!=}} \begin{vmatrix} 1 & 2 & 2 & 4 \\ 0 & 6 & 2 & 10 \\ 0 & -3 & -5 & -10 \\ 0 & -2 & 6 & 6 \end{vmatrix}$$

cof. expand along 1st col.

$$1 \cdot \begin{vmatrix} 6 & 2 & 10 \\ -3 & -5 & -10 \\ -2 & 6 & 6 \end{vmatrix} \underset{\substack{R2 \to (-1)R2 \\ R3 \to \frac{1}{2}R3}}{\overset{R1 \to \frac{1}{2}R1}{=\!=}} 2 \cdot (-1) \cdot 2 \begin{vmatrix} 3 & 1 & 5 \\ 3 & 5 & 10 \\ -1 & 3 & 3 \end{vmatrix}$$

$$= (-4)\left(3 \cdot 5 \cdot 3 + 3 \cdot 3 \cdot 5 + 1 \cdot 10 \cdot (-1) - 5 \cdot 5 \cdot (-1) - 3 \cdot 10 \cdot 3 - 3 \cdot 3 \cdot 1\right)$$

$$= (-4)(45 + 45 - 10 + 25 - 90 - 9) = -4 \cdot 6 = -24$$

3. (10 points) In each of the following, determine whether the subset S is a subspace of the given vector space V:

i) $V = \mathbb{R}^4$ and $S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 x_4 = 0\}$;

$(1,0,0,0) \in S$ as $1 \cdot 0 = 0$

$(0,0,0,1) \in S$ as $0 \cdot 1 = 0$

$(1,0,0,0) + (0,0,0,1) = (1,0,0,1) \notin S$ as $1 \cdot 1 \neq 0$

Hence S not a subspace.

ii) $V = M_{2\times 2}(\mathbb{R})$ and $S = \{A \in M_{2\times 2}(\mathbb{R}) \mid A = 2A^T\}$.

$O_{2\times 2} \in S$ as $2\, O_{2\times 2}^T = 2 \cdot O_{2\times 2} = O_{2\times 2}$. Thus S not empty

Take $A_1, A_2 \in S$. Then $A_1 = 2A_1^T$ and $A_2 = 2A_2^T$.

$2(A_1 + A_2)^T = 2(A_1^T + A_2^T) = 2A_1^T + 2A_2^T = A_1 + A_2$ so $A_1 + A_2 \in S$

Closed under addition

Let k be a scalar. Then

$2(kA_1)^T = 2kA_1^T = k(2A_1^T) = kA_1$ so $kA_1 \in S$

Closed under scalar multiplication

Hence S a subspace.

4. **(10 points)** Using the Wronskian, determine whether or not the functions

$$f_1(x) = \sin x\,,\ f_2(x) = \sin 2x\,,\ f_3(x) = e^x$$

are linearly independent on $\mathbb{R}$.

$$W(x) = \begin{vmatrix} \sin(x) & \sin(2x) & e^x \\ \cos(x) & 2\cos(2x) & e^x \\ -\sin(x) & -4\sin(2x) & e^x \end{vmatrix} = e^x \begin{vmatrix} \sin(x) & \sin(2x) & 1 \\ \cos(x) & 2\cos(2x) & 1 \\ -\sin(x) & -4\sin(2x) & 1 \end{vmatrix}$$

$$= e^x \big(2\sin(x)\cos(2x) - 4\sin(2x)\cos(x) - \sin(2x)\sin(x)$$
$$+ 2\sin(x)\cos(2x) - \sin(2x)\cos(x) + 4\sin(2x)\sin(x)\big)$$

$$= e^x \big(4\sin(x)\cos(2x) + 3\sin(2x)\sin(x) - 5\sin(2x)\cos(x)\big)$$

$W(0) = 0$ can not conclude anything

$W(\frac{\pi}{2}) = e^{\pi/2}(4\cdot 1\cdot(-1) + 3\cdot 0\cdot 1 - 5\cdot 0\cdot 0) = -4e^{\pi/2} \neq 0$

Hence $\{f_1, f_2, f_3\}$ lin. indep.

5. (10 points) Find a subset of

$$S = \left\{ \begin{pmatrix} 3 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$$

that forms a basis for the subspace of $\mathbb{R}^4$ generated by S, i.e., span S.

Identify S with colspace(A) where $A = \begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 1 & 3 & 4 \end{bmatrix}$

$$A \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 1 & 3 & 4 \end{bmatrix} \xrightarrow[\substack{R3 \to R3 - 2R1 \\ R4 \to R4 - 2R1}]{R2 \to R2 - 2R1} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & -1 & 1 & 6 \end{bmatrix} \xrightarrow{R4 \to R4 - R3}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow[\substack{R3 \to R3/5 \\ R4 \to R4/2}]{R_2 \to (-1) R_2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R4 \to R4 - R3} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

columns 1, 2 and 4 in original A form a basis.

Thus $\left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$ form a basis

for span S.

6. (10 points) For the matrix

$$A = \begin{bmatrix} 3 & -1 & -3 & 1 & 2 \\ 5 & -2 & -6 & 3 & 6 \\ 2 & 0 & -1 & 0 & -1 \end{bmatrix},$$

find a basis and the dimension for nullspace(A).

$$\left[\begin{array}{ccccc|c} 3 & -1 & -3 & 1 & 2 & 0 \\ 5 & -2 & -6 & 3 & 6 & 0 \\ 2 & 0 & -1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \to R1 - R3} \left[\begin{array}{ccccc|c} 1 & -1 & -2 & 1 & 3 & 0 \\ 5 & -2 & -6 & 3 & 6 & 0 \\ 2 & 0 & -1 & 0 & -1 & 0 \end{array}\right] \xrightarrow[R3 \to R3 - 2R1]{R_2 \to R2 - 5R1}$$

$$\to \left[\begin{array}{ccccc|c} 1 & -1 & -2 & 1 & 3 & 0 \\ 0 & 3 & 4 & -2 & -9 & 0 \\ 0 & 2 & 3 & -2 & -7 & 0 \end{array}\right] \xrightarrow{R2 \to R2 - R3} \left[\begin{array}{ccccc|c} 1 & -1 & -2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & -2 & -7 & 0 \end{array}\right] \xrightarrow{R3 \to R3 - 2R2}$$

$$\to \left[\begin{array}{ccccc|c} 1 & -1 & -2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2 & -3 & 0 \end{array}\right]$$

x_1, x_2, x_3 bounded variables

$x_4 = \alpha$, $x_5 = \beta$ free variables

$$x_3 = 2x_4 + 3x_5 = 2\alpha + 3\beta$$

$$x_2 = -x_3 + 2x_5 = -2\alpha - 3\beta + 2\beta = -2\alpha - \beta$$

$$x_1 = x_2 + 2x_3 - x_4 - 3x_5 = -2\alpha - \beta + 4\alpha + 6\beta - \alpha - 3\beta = \alpha + 2\beta$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \alpha + 2\beta \\ -2\alpha - \beta \\ 2\alpha + 3\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$ basis and dimension is 2