MTH 165: Linear Algebra with Differential Equations

Final Exam

May 5, 2014

NAME (please print legibly): ________________________________

Your University ID Number: ________________________________

Indicate your instructor with a check in the box:

Friedmann	MW 16:50 - 18:05	
Karapetyan	MW 14:00-15:15	
Petridis	MWF 10:00 - 10:50	

- You have 3 hours to work on this exam.
- No calculators, cell phones, other electronic devices, books, or notes are allowed.
- Show all your work and simplify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You are responsible for checking that this exam has all 11 pages.

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	

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1. (10 points) Find the explicit solution to the following initial value problem showing all your work

$$y' - x\,y^2 = x\,,\ y(0) = 1\,.$$

The equation is not linear. So we proceed by separating variables and integrating.

$$y' = x\,(y^2+1) \implies \frac{dy}{y^2+1} = x\,dx \implies \int \frac{dy}{y^2+1} = \int x\,dx.$$

So

$$\arctan(y) = \frac{x^2}{2} + c.$$

Setting $x = 0$ gives

$$c = \arctan(y(0)) = \arctan(1) = \frac{\pi}{4}.$$

Therefore

$$y = \tan\left(\frac{x^2}{2} + \frac{\pi}{4}\right).$$

2. (10 points) (i) Let A be an invertible $n \times n$. Is it true that the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$? It it is, carefully explain why. If it is not provide an explicit example that disproves the claim.

Yes.

Multiplying both sides of the vector equation by A^{-1} on the left gives:

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

(ii) Is there a system of three distinct linear equations in two unknowns that has a unique solution? If there is, provide an explicit example. If there is not, explain carefully why this is the case.

Yes.

For example

$$\begin{cases} x_1 + x_2 = 0 \\ 2x_1 + x_2 = 0 \\ x_1 + 2x_2 = 0 \end{cases}$$

only has the trivial solution $x = y = z = 0$.

(iii) Is there a system of two distinct linear equations in three unknowns that has a unique solution? If there is, provide an explicit example. If there is not, carefully explain why this is the case.

No.

The rank of matrix is at most 2 and so cannot equal the number of unknowns.

A less precise way to phrase this to note that " after row reduction, there must be a free variable" – there are simply not enough equations so that every unknown corresponds to a column with a leading 1.

3. (10 points) Let A and B be the following matrices

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \text{ and } B = -2 \begin{bmatrix} a_1+c_1 & c_1 & -1 & b_1 \\ a_2+c_2 & c_2 & 3 & b_2 \\ 0 & 0 & 2 & 0 \\ a_3+c_3 & c_3 & 7 & b_3 \end{bmatrix}.$$

Find $\det(B)$ in terms of $\det(A)$ showing all your work.

A standard property of the determinant gives

$$\det(B) = (-2)^4 \begin{vmatrix} a_1+c_1 & c_1 & -1 & b_1 \\ a_2+c_2 & c_2 & 3 & b_2 \\ 0 & 0 & 2 & 0 \\ a_3+c_3 & c_3 & 7 & b_3 \end{vmatrix}.$$

Expanding along the third row gives

$$\det(B) = 16 \cdot 2 \begin{vmatrix} a_1+c_1 & c_1 & b_1 \\ a_2+c_2 & c_2 & b_2 \\ a_3+c_3 & c_3 & b_3 \end{vmatrix}.$$

A matrix and its transpose have the same determinant and therefore

$$\det(B) = 32 \begin{vmatrix} a_1+c_1 & a_2+c_2 & a_3+c_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Interchanging the bottom two rows gives

$$\det(B) = -32 \begin{vmatrix} a_1+c_1 & a_2+c_2 & a_3+c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Subtracting the bottom row from the top row gives Interchanging the bottom two rows gives

$$\det(B) = -32 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Therefore we have

$$\det(B) = -32 \det(A).$$

4. (10 points) Let $M_2(\mathbb{R})$ be the vector space of 2×2 matrices with real components, 0_2 be the zero 2×2 matrix, B be the following matrix

$$B = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix},$$

and S be the following subset of $M_2(\mathbb{R})$

$$S = \{A \in M_2(\mathbb{R}) : AB = 0_2\}.$$

Prove that S is a subspace of $M_2(\mathbb{R})$; find a basis for it; and determine its dimension.

S is a subset because:

- it contains the zero matrix: $0_2 \in S$ because $0_2 B = 0_2$.

- it is closed under addition: Let $A_1, A_2 \in S$, then $(A_1 + A_2) \in S$ because

$$(A_1 + A_2)B = A_1 B + A_2 B = 0_2 + 0_2 = 0_2.$$

- it is closed under scalar multiplication: Let $A \in S$ and $\lambda \in \mathbb{R}$ then $(\lambda A) \in S$ because

$$(\lambda A)B = \lambda(AB) = \lambda\, 0_2 = 0_2.$$

To find a basis we note that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is equivalent to the system

$$\begin{cases} 2a - b = 0 \\ 2c - d = 0 \end{cases}.$$

So $b = 2a$ and $d = 2c$ and therefore every element in S has the form

$$\begin{bmatrix} a & 2a \\ c & 2c \end{bmatrix} = a \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}.$$

The set of the two matrices above is linearly independent (none is a scalar multiple of the other) and therefore S has basis

$$\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \right\}.$$

The dimension of S is the number of element in the basis.

$$\dim(S) = 2.$$

5. (10 points) Let A be the following matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 4 \end{bmatrix}.$$

Define a linear transformation T by $T\mathbf{x} = A\mathbf{x}$.

(i) The kernel of T is a subspace of $\mathbb{R}^d$ for what value of d?

$\underline{d = 3}$.

(ii) Find a basis for the kernel of T showing all your work.

The reduced row-echelon form of A is

$$E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

x_3 is a free variable as it does not correspond to a leading 1. It follows from back substitution that every vector in the nullspace is of the form $\begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix}$.

In other words Ker(T) has basis

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(iii) The range (or image) of T is a subspace of $\mathbb{R}^d$ for what value of d?

$\underline{\ d = 2}$.

(iv) What is the dimension of the range of T? Justify your answer.

T is linear. The Rank-Nullity Theorem gives

$$\dim(\text{Rng}(T)) = \dim(\text{domain of } T) - \dim(\text{Ker}(T)) = 3 - 1 = 2.$$

6. (10 points) Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(i) Determine its eigenvalues and their (algebraic) multiplicities showing all your work.

The characteristic polynomial of A is

$$\det(A-tI) = \begin{vmatrix} 2-t & -1 & 0 \\ 1 & -t & 0 \\ 0 & 0 & 3-t \end{vmatrix} = (3-t)\begin{vmatrix} 2-t & -1 \\ 1 & -t \end{vmatrix} = (3-t)(t^2-2t+1) = -(t-3)(t-1)^2.$$

Therefore the eigenvalues are: 1 with multiplicity 2 and 3 with multiplicity 1.

(ii) Find a basis for each eigenspace showing all your work.

For $\lambda = 1$ the reduced row-echelon form of $A - I$ is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. So $x_3 = 0$, x_2 is a free variable and $x_1 = x_2$. Therefore a basis for the eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

For $\lambda = 3$ the reduced row-echelon form of $A - I$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So $x_1 = x_2 = 0$ and x_3 is a free variable therefore a basis for the eigenspace is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(iii) Conclude, with explanation, whether A is a defective or non-defective matrix.

A is defective as it is a 3×3 matrix and the sum of the dimensions of its eigenspaces is $1 + 1 < 3$.

Equivalently A has only two linearly independent eigenvectors.

It is also enough to note that the geometric multiplicity of $\lambda = 1$ is not equal to its algebraic multiplicity.

7. (10 points) Find the general solution to each of the following differential equations showing all your work.

(i) $t^2\, y'' \;-\; 8\, t\, y' \;+\; 20\, y \;=\; 0\,.$

[Hint: try powers of t as solutions.]

We follow the hint and try

$$y(t) = t^r.$$

Differentiating and substituting gives

$$t^r\,(r(r-1)-8r+20) = 0 \implies r(r-1)-8r+20 = 0 \implies r^2-9r+20 = 0 \implies (r-4)(r-5) = 0.$$

So

$$y_1 = t^4 \text{ and } y_2 = t^5$$

are two linearly independent solutions (none is a multiple of another).

Therefore they form a basis for the 2-dimensional vector space of solutions. The general solution is

$$y(x) = c_1\, t^4 + c_2\, t^5,$$

where c_1, c_2 are arbitrary constants.

(ii) $y^{(4)} \;+\; 4\, y^{(3)} \;+\; 10\, y'' \;+\; 12\, y' \;+\; 9\, y \;=\; 0\,.$

[Hint: $r^4 + 4r^3 + 10r^2 + 12r + 9 = (r^2 + 2r + 3)^2$.]

The auxiliary polynomial is

$$p(r) = (r^2 + 2r + 3)^2.$$

Its roots are

$$r = -1 \pm i\sqrt{2}.$$

Both have multiplicity 2 and so the general solution is

$$y = c_1\, e^{-t}\, \cos(\sqrt{2}\, t) + c_2\, e^{-t}\, \sin(\sqrt{2}\, t) + c_3\, t\, e^{-t}\, \cos(\sqrt{2}\, t) + c_4\, t\, e^{-t}\, \sin(\sqrt{2}\, t),$$

where $c_1, \ldots, c_4$ are arbitrary constants.

8. (10 points) Find the general solution to the following differential equation showing all your work.

$$y''' - y'' = 12\,x^2 \,.$$

Note that

$$y = y_c + y_p \,.$$

y_c, the general solution to the homogenous equation, is determined by the roots of the auxiliary polynomial $p(r) = r^3 - r^2 = r^2(r-1)$.

They are $r = 0$ with multiplicity 2 and $r = 1$ with multiplicity 1. Therefore

$$y_c = c_1\, e^{0x} + c_2\, e^{0x}\, x + c_3\, e^x = c_1 + c_2\, x + c_3\, e^x \,.$$

For the particular solution y_p we try

$$y_p = A\,x^2 + B\,x^3 + C\,x^4 \,.$$

This is because the $12x^2$ terms on the right hand side corresponds to the $r = 0$ root. "Normally" one would try $y_p = A + Bx + Cx^2$, but because the multiplicity of $r = 0$ is 2, one must multiply this expression by x^2.

Substituting in the differential equation gives

$$12x^2 = (0 + 6B + 24Cx) - (2A + 6Bx + 12Cx^2) = (6B - 24A) + (24C - 6B) - 12Cx^2 \,.$$

Equating coefficients of powers of x gives

$$\begin{cases} 6B - 2A = 0 \\ 24C - 6B = 0 \\ -12C = 12 \end{cases} \implies \begin{cases} C = -1 \\ B = 4C = -4 \\ A = 3B = -12 \end{cases} \,.$$

this gives

$$y_p = -12x^2 - 4x^3 - x^4 \,.$$

Finally,

$$y = c_1 + c_2\,x + c_3\,e^x - 12\,x^2 - 4\,x^3 - x^4 \,.$$

9. (10 points) A spring with spring constant 8 N/m is loaded with a 2 kg mass and allowed to reach equilibrium. It is then displaced and released. Suppose that after $\pi/2$ seconds the mass is 1 m below the equilibrium position and moving upward with speed $2\sqrt{3}$ m/s.

Find the equation of motion of the displacement $y(t)$ from the equilibrium position, the amplitude, and the phase. Neglect friction. Show all your work.

Taking the downward direction as positive and applying Newton's law, gives the following initial value problem

$$2\,y'' + 8\,y = 0\,,\ y(\pi/2) = 1\,,\ y'(\pi/2) = -2\sqrt{3}\,.$$

Notes: There is no y' as there is no friction; the equation is homogeneous as there is no external force; $y(\pi/2) = 1$ as the mass is below the equilibrium position; $y'(0) = -2\sqrt{3}$ as the mass is moving upward.

The auxiliary polynomial is $p(r) = r^2 + 4$ with roots $\pm 2i$. Therefore

$$y = c_1\,\cos(2\,t) + c_2\,\sin(2\,t).$$

Setting $t = \pi/2$ gives

$$1 = c_1\,\cos(\pi) + c_2\sin(\pi) = -c_1.$$

So

$$y = -\cos(2\,t) + c_2\sin(2\,t)\ \text{and}\ y' = 2\,\sin(2\,t) + 2\,c_2\,\cos(2\,t).$$

Setting $t = \pi/2$ gives

$$-2\sqrt{3} = 2\,\sin(\pi) + 2\,c_2\,\cos(\pi) = -2\,c_2.$$

Therefore

$$y = -\cos(2\,t) + \sqrt{3}\,\sin(2\,t).$$

Using (standard trigonometric identities) or the formulas

$$A_0 = \sqrt{c_1^2 + c_2^2} = \sqrt{1+3} = 2\ \text{and}\ \phi = \arctan\left(\frac{c_2}{c_1}\right) = \arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

gives

$$y(t) = -\cos(2t) + \sqrt{3}\,\sin(2t) = 2\,\cos\left(2t + \frac{\pi}{3}\right).$$

The amplitude is 2 and the phase is $-\frac{\pi}{3}$.

10. (10 points) Solve the following initial value problem showing all your work.

$$\begin{cases} x_1' = x_1 + 3x_2, \quad x_1(0) = 2 \\ x_2' = -3x_1 + x_2, \quad x_2(0) = 3 \end{cases}.$$

Written in terms of linear maps the system takes the form

$$\begin{cases} (D - I)x_1 = 3x_2 \\ (D - I)x_2 = -3x_1 \end{cases}.$$

Applying the linear differential operator $(D - I)$ to the top equation gives

$$(D - I)(D - I) = 3(D - I)x_2 \overset{\text{2nd eqn}}{=} -9x_1 \iff (D^2 - 2D + 10I)x_1 = 0.$$

The auxiliary polynomial is $p(r) = r^2 - 2r + 10$. Its roots are $r = 1 \pm 3\,i$. Therefore

$$x_1 = c_1\, e^t\, \cos(3t) + c_2\, e^t \sin(3t).$$

Substituting in the top equation gives

$$3x_2 = x_1' - x_1 = -3\, c_1\, e^t\, \sin(3t) + 3\, c_2\, \cos(3t) \implies x_2 = -c_1\, e^t\, \sin(3t) + c_2\, \cos(3t)\,.$$

The initial conditions give $2 = x_1(0) = c_1$ and $3 = x_2(0) = c_2$ so the solution is

$$\begin{cases} x_1 = 2\, e^t\, \cos(3t) + 3\, e^t\, \sin(3t) \\ x_2 = 3\, e^t\, \cos(3t) - 2\, e^t\, \sin(3t) \end{cases}.$$

Note: It is OK to use eigenvalues/eigenvectors and real valued functions.

Can in fact discard the second row of the matrix $A - \lambda I$ because the rank of the matrix cannot be 2 (else the nullspace would be trivial and λ would not be an eigenvalue) and so the second row is a multiple of the first.