MTH 165: Linear Algebra with Differential Equations

First Midterm
February 25, 2014

NAME (please print legibly): Solutions
Your University ID Number:
Indicate your instructor with a check in the box:

Freidmann	MW 16:50 - 18:05	
Karapetyan	MW 14:00-15:15	
Petridis	MWF 10:00 - 10:50	

- You have 75 minutes to work on this exam.
- No calculators, cell phones, other electronic devices, books, or notes are allowed during this exam.
- Show all your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- In your answers, you do not need to simplify arithmetic expressions like $\sqrt{\sin(5)}$. However, you must simplify logarithmic expressions like $\ln e$, trigonometric expressions like $\sin \pi$, and expressions like $-\frac{-1}{1/x^{-1}}$ as much as possible.
- You are responsible for checking that this exam has all 7 pages.

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
TOTAL	60	

By taking this exam, you are acknowledging that the following is prohibited by the College's Honesty Policy: Obtaining an examination prior to its administration. Using unauthorized aid during an examination or having such aid visible to you during an examination. Knowingly assisting someone else during an examination or not keeping your work adequately protected from copying by another.

1. (10 points) Find the general solution of the differential equation on the interval $(0,2)$

$$x^3\frac{dy}{dx}+3x^2y+\frac{2}{4-x^2}=0\,.$$

Give your answer in explicit form.

$$x^3y'+3x^2y=-\frac{2}{4-x^2}=-\frac{1}{2}\left[\frac{1}{2+x}+\frac{1}{2-x}\right]$$

$\text{LHS}=\frac{d}{dx}(x^3\cdot y)$ so $[x^3y]'=-\frac{1}{2}\left[\frac{1}{2+x}+\frac{1}{2-x}\right]$

Integrating both sides gives:

$$x^3y=\int-\frac{1}{2}\left[\frac{1}{2+x}+\frac{1}{2-x}\right]dx=\frac{1}{2}\left[-\ln(|2+x|)+\ln|(2-x)|\right]+C$$

on $(0,2)$ both $2+x$, $2-x>0$ so

$$x^3y=\frac{1}{2}\left(-\ln(2+x)+\ln(2-x)\right)+C$$

So the explicit general solution is

$$y=\frac{\ln\left(\frac{2-x}{2+x}\right)}{2x^3}+\frac{C_1}{x^3}\qquad\left[C_1=\frac{C}{2}\right]$$

2. (10 points) Solve the initial value problem on the interval $[e^{-1}, \infty)$.

$$x \ln(x) \sin(y) \frac{dy}{dx} - \sin\left(\frac{\pi}{2} - y\right) = 0, \quad y(e^{-1}) = -\pi.$$

Give your solution in implicit form.

$\sin\left(\frac{\pi}{2} - y\right) = \cos(y)$ and so

$x \ln(x) \sin(y) \frac{dy}{dx} = \cos(y)$. In differential form:

$\frac{\sin(y)}{\cos(y)} dy = \frac{dx}{x \ln(x)}$. Integrating

$\int \frac{\sin(y)}{\cos(y)} dy = \int \frac{dx}{x \ln(x)}$. Setting $u = \cos(y)$, $v = \ln(x)$:

$$\int -\frac{du}{u} = \int \frac{dv}{v} \Rightarrow \ln\left(\left|\frac{1}{u}\right|\right) = \ln(|v|) + c$$

$$\Rightarrow \left|\frac{1}{u}\right| = |v| \cdot e^{c}$$

$$\Rightarrow \left|\frac{1}{u}\right| = |v| \cdot c_1 \qquad [c_1 = e^{c}]$$

$$\Rightarrow \frac{1}{u} = v \cdot c_2 \qquad [c_2 \text{ depends on abs. values}]$$

$$\Rightarrow \frac{1}{\cos(y)} = c_2 \ln(x)$$

Setting $x = e^{-1}$ gives: $\frac{1}{\cos(-\pi)} = c_2 \ln(e^{-1})$ So

$$-1 = c_2 (-1) \Rightarrow \boxed{c_2 = 1}$$

The implicit solution is $\boxed{\cos(y) = \frac{1}{\ln(x)}}$

3. (10 points) A large tank initially contains 1 L of solution in which there is dissolved 10 g of salt. A solution containing te^{-t} g/L of salt, where t is the time in minutes from the initial state, flows into the tank at a rate of 1 L/min. Through another faucet pure water flows into the tank at a rate of 1 L/min. The well-stirred mixture flows out of the tank at a rate of 2 L/min. Find $c(t)$, the concentration of salt in the well-stirred mixture after t minutes. What value does $c(t)$ tend to, when $t \to \infty$?

$V(t)$ = volume after t seconds. $\frac{dV}{dt} = (\text{rate in}) - (\text{rate out})$
$= (1+1) - 2 = 0.$

So $V(t) = V(0) = 1$ L

$A(t)$ = amount of salt at time t. $\frac{dA}{dt} = \left(\text{rate} \times \text{concentration in}\right) - \left(\text{rate} \times \text{concentration out}\right)$

$= (1 \cdot te^{-t} + 1 \cdot 0) - 2 \cdot \frac{A(t)}{V(t)}$

$= te^{-t} - 2A.$

So $A' + 2A = te^{-t}$.

Integrating factor is $e^{\int 2\,dt} = e^{2t}$. Multiplying:

$e^{2t} \cdot A' + 2e^{2t} A = te^{t} \Rightarrow \left(e^{2t} A\right)' = te^{t}$

Integrating: $e^{2t} A = \int te^{t}\,dt$.

RHS: Let $u = t$ $dv = e^t\,dt$ then $\int te^t = t \cdot e^t - \int 1 \cdot e^t = te^t - e^t + C$
i.e. $v = e^t$

Therefore, $e^{2t} A = te^t - e^t + C$

Setting $t=0$: $e^{2 \cdot 0} \cdot 10 = 0 - 1 + C$ so $C = 11$

That is $A(t) = te^{-t} - e^{-t} + 11e^{-2t}$

$c(t) = \frac{A(t)}{V(t)} = \frac{A(t)}{1}$ so $\boxed{c(t) = te^{-t} - e^{-t} + 11e^{-2t}}$

As $t \to \infty$, $c(t) \to 0 - 0 + 0 = 0$.

This makes sense as $te^{-t} \to 0$ so in the limiting case "almost pure" water enters the tank.

4. (10 points) Solve the following system of linear equations or show it is inconsistent.

$$\begin{cases} 3x_1 - 2x_2 + 12x_3 + 20x_4 = -1 \\ 2x_1 - x_2 + 8x_3 + 14x_4 = 1 \\ 2x_1 + 8x_3 + 9x_4 = -1 \end{cases}.$$

$$A^\# = \left[\begin{array}{cccc|c} 3 & -2 & 12 & 20 & -1 \\ 2 & -1 & 8 & 14 & 1 \\ 2 & 0 & 8 & 9 & -1 \end{array}\right] \overset{A_{21}(-1)}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 4 & 6 & -2 \\ 2 & -1 & 8 & 14 & 1 \\ 2 & 0 & 8 & 9 & -1 \end{array}\right]$$

$$\underset{A_{13}(-2)}{\overset{A_{12}(-2)}{\sim}} \left[\begin{array}{cccc|c} 1 & -1 & 4 & 6 & -2 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 2 & 0 & -3 & 3 \end{array}\right] \overset{A_{23}(-2)}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 4 & 6 & -2 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 0 & -7 & -7 \end{array}\right]$$

$$\overset{M_3(-\frac{1}{7})}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 4 & 6 & -2 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

$\text{rank}(A) = \text{rank}(A^\#) = 3 < 4$ = number of unknowns.
So there is $4-3=1$ free variable.
Chose as the free variable x_3 as the third column has no leading 1. $\boxed{x_3 = t}$

Back substitution: 3rd row: $x_4 = 1$

2nd row: $x_2 = 5 - 2x_4 = 5 - 2 = 3$.

1st row: $x_1 = x_2 - 4x_3 - 6x_4 - 2$
$= 3 - 4t - 6 - 2$
$= -5 - 4t$.

So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5-4t \\ 3 \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Check: Setting $t=0$ gives a solution

5. (10 points) Solve the following system of linear equations or show it is inconsistent.

$$\begin{cases} 3x_1 + 2x_2 - x_3 + 5x_4 = 6 \\ 4x_1 + 5x_2 + x_3 + 3x_4 = 2 \\ x_1 - 4x_2 - 5x_3 + 9x_4 = -4 \end{cases}.$$

$$A^{\#} = \left[\begin{array}{cccc|c} 3 & 2 & -1 & 5 & 6 \\ 4 & 5 & 1 & 3 & 2 \\ 1 & -4 & -5 & 9 & -4 \end{array}\right] \overset{P_{13}}{\sim} \left[\begin{array}{cccc|c} 1 & -4 & -5 & 9 & -4 \\ 3 & 2 & -1 & 5 & 6 \\ 4 & 5 & 1 & 3 & 2 \end{array}\right]$$

$$\underset{A_{13}(-4)}{\overset{A_{12}(-3)}{\sim}} \left[\begin{array}{cccc|c} 1 & -4 & -5 & 9 & -4 \\ 0 & 14 & 14 & -22 & 18 \\ 0 & 21 & 21 & -33 & 18 \end{array}\right] \overset{A_{23}(-\frac{3}{2})}{\sim} \left[\begin{array}{cccc|c} 1 & -4 & -5 & 9 & -4 \\ 0 & 14 & 14 & -22 & 18 \\ 0 & 0 & 0 & 0 & -9 \end{array}\right]$$

The system is inconsistent as the last row suggest $0x_1 + 0x_2 + 0x_3 + 0x_4 = -9$!

Or $\text{rank}(A) = 2 < 3 = \text{rank}(A^{\#})$.

6. (10 points) Let A be the 3×3 matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$. Find each of A^{-1} and $A^{-1}A^T$ or show that they do not exist. Use your answer to solve the following linear system of equations: $\begin{cases} x_1 + x_3 = 1 \\ x_2 = 1 \\ x_1 + x_2 + 2x_3 = 1 \end{cases}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \overset{A_{13}(-1)}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \overset{A_{23}(-1)}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$\overset{A_{31}(-1)}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \text{ so } \boxed{A^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}$$

Check:

$$A A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}!$$

$$A^{-1}A^T = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix} = A^{-1}A^T}$$

The system when written in matrix vector form becomes

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Multiplying on the left by A^{-1}:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

Check! ✓