MTH 165: Linear Algebra with Differential Equations

Second Midterm
April 1, 2014

NAME (please print legibly): Solutions
Your University ID Number:
Indicate your instructor with a check in the box:

Friedmann	MW 16:50 - 18:05	
Karapetyan	MW 14:00-15:15	
Petridis	MWF 10:00 - 10:50	

- You have 75 minutes to work on this exam.
- No calculators, cell phones, other electronic devices, books, or notes are allowed during this exam.
- Show all your work and justify your answers, especially when instructed to do so. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You are responsible for checking that this exam has all 7 pages.

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
TOTAL	60	

By taking this exam, you are acknowledging that the following is prohibited by the College's Honesty Policy: Obtaining an examination prior to its administration. Using unauthorized aid during an examination or having such aid visible to you during an examination. Knowingly assisting someone else during an examination or not keeping your work adequately protected from copying by another.

1. (10 points) Let A be the 3×3 matrix

$$A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

with rows $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$. Evaluate in terms of $\det(A)$ the determinant of each of the following matrices, showing all your work.

(i) $B = \begin{bmatrix} 3\mathbf{r}_1 + 2\mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_2 \end{bmatrix}$.

$$\det(B) = \begin{vmatrix} 3\vec{r_1} + 2\vec{r_2} \\ \vec{r_3} \\ \vec{r_2} \end{vmatrix} = -\begin{vmatrix} 3\vec{r_1} + 2\vec{r_2} \\ \vec{r_2} \\ \vec{r_3} \end{vmatrix}$$

$$= -\begin{vmatrix} 3\vec{r_1} \\ \vec{r_2} \\ \vec{r_3} \end{vmatrix} = -3\begin{vmatrix} \vec{r_1} \\ \vec{r_2} \\ \vec{r_3} \end{vmatrix} = -3 \det(A).$$

(ii) $C = 2A^T A^2$.

$$\det(C) = \det(2A^T A^2)$$
$$= 2^3 \det(A^T A^2)$$
$$= 2^3 \det(A^T) \det(A^2)$$
$$= 2^3 \det(A) \det(A)^2$$
$$= 2^3 \det(A)^3.$$

2. (10 points) Determine whether each of the following subsets of $M_2(\mathbb{R})$, the vector space of 2×2 matrices with real components, is a subspace. Justify your answers.

(i) $U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a^2 - b^2 = 0 \right\}$.

No, $\mathcal{U}$ is not a subspace. It is not closed addition.
For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ belong to $\mathcal{U}$.
However, $A + B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ doesn't.

(ii) $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 3b + 2d = 0 \right\}$.

Yes, V is a subspace!
(i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$ as $3 \cdot 0 + 2 \cdot 0 = 0$
(ii) Closure under addition: Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in V$.
Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} \in V$
because $3(b+b') + 2(d+d') = (3b+2d) + (3b'+2d')$
$= 0 + 0 = 0$.
(iii) Closure under scalar multiplication.
Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ and $\lambda \in \mathbb{R}$.
Then $\lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix} \in V$
because $3(\lambda b) + 2(\lambda d) = \lambda(3b + 2d) = \lambda \cdot 0 = 0$.

3. (10 points) Determine whether each of the following sets of vectors spans $\mathbb{R}^3$. Justify your answers.

(i) $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

It spans $\mathbb{R}^3$ because it contains three vectors and $\det\left(\begin{bmatrix} 1 & 1 & 1 \\ -1 & 7 & 0 \\ -1 & 3 & 0 \end{bmatrix}\right) = 1 \cdot \det\left(\begin{bmatrix} -1 & 7 \\ -1 & 3 \end{bmatrix}\right) = 4 \neq 0$.

[Alternatively, row reduce to get that $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 7 & 0 \\ -1 & 3 & 0 \end{bmatrix}$ has rank 3 and so the span of the vectors has dimension 3]

(ii) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \right\}$.

It does not span $\mathbb{R}^3$ because

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ -1 & 1 & 2 & -1 \\ 2 & 1 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 3 & 3 & 2 \\ 0 & -3 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore by the column space-basis-theorem the span of the set has basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ and therefore has dimension 2.

4. (10 points) Determine whether each of the following sets of functions is linearly dependent or linearly independent (over $\mathbb{R}$). Justify your answers.

(i) $\{\cos(2x), \sin(x)^2, \cos(x)^2, 1\}$.

The set is linearly dependent as

$$1 \cdot \cos(2x) - 1 \cdot \sin(x)^2 - 1 \cos(x)^2 = O(t) \leftarrow$$ "Oh of t" the zero function.

Therefore there is a non-trivial linear combination of the set elements giving the zero function.

(ii) $\{x, x^2, e^{2x}\}$.

The set is linearly independent. We use the Wronskian:

$$W(x) = \begin{vmatrix} x & x^2 & e^{2x} \\ x' & (x^2)' & (e^{2x})' \\ x'' & (x^2)'' & (e^{2x})'' \end{vmatrix} = \begin{vmatrix} x & x^2 & e^{2x} \\ 1 & 2x & 2e^{2x} \\ 0 & 2 & 4e^{2x} \end{vmatrix}.$$

$$W(0) = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{vmatrix} \underset{\substack{\uparrow \\ \text{first} \\ \text{row}}}{=} 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 \neq 0.$$

$W(2) \neq 0$ so the set is linearly independent.

5. (10 points) Find a basis for the following subspace of P_3, the vector space of all real polynomials of degree at most 3:

$$W = \{p(t) \in P_3 : p''(t) = \text{ a constant}\}.$$

What is the dimension of W? Justify your answers. Note that you are not asked to prove that W is a subspace.

Let $p(t) \in W$. We determine its "form".

$p''(t) = c$ integrating gives

$p'(t) = ct + b$ integrating again gives

$p(t) = \frac{ct^2}{2} + bt + a$

So $W = \{ \frac{ct^2}{2} + bt + a : a, b, c \in \mathbb{R} \}$

$\{1, t, t^2\}$ is a basis for W because:

→ it spans W: $p(t) = \frac{c}{2} \cdot t^2 + b \cdot t + a \cdot 1$

→ it is linearly independent, being a subset of the standard basis or via the Wronskian

$W(x) = \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$ for all x.

Therefore $\dim(W) = \#$ elements in the basis $= 3$.

Note: The dimension of W is the size of the basis. Bases do not have dimension.

6. (10 points) Let A be the 3×4 matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ -1 & 1 & -2 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}.$$

Answer each of the following questions showing all your work.

(i) Find a row-echelon form for A.

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ -1 & 1 & -2 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = E$$

(ii) Find a basis for the nullspace of A.

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \text{nullspace}(A) \iff \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \text{nullspace}(E)$$

$$\iff \begin{cases} z = 0 & \leftarrow \text{last row of } E \\ y = z = 0 & \leftarrow \text{middle row of } E \\ x = w \end{cases}$$

Set w to be the free variable as the fourth column in E has no leading 1. $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = w \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ So $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the nullspace

(iii) Find a basis for the column space of A.

From the theorem!

$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} \right\}$ is a basis for the columnspace of A because the 1^{st}, 2^{nd}, 3^{rd} columns of E have leading 1s.

(iv) Find a basis for the row space of A.

$\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the row space as all rows of E are non-zero