MTH 165: Linear Algebra with Differential Equations

First Midterm ANSWERS

March 4, 2015

1. (10 points) Solve the following initial value problem on the interval $[0, \infty)$

$$2y(1+x^3)y' - x^2y^2 = ax^2, \; y(0) = a,$$

where $a > 0$ is a positive constant. Give your answer in implicit form.

This is a separable differential equation.

$$2yy'(1+x^3) = x^2(y^2+a) \iff \frac{2ydy}{y^2+a} = \frac{x^2dx}{1+x^3}.$$

Integrating gives

$$\int \frac{2ydy}{y^2+a} = \int \frac{x^2dx}{1+x^3}.$$

which implies

$$\ln(|y^2+a|) = \frac{1}{3}\ln(|1+x^3|) + C.$$

All quantities are positive so the absolute values are eliminated. Setting $x = 0$ gives $C = \ln(a^2+a)$. Therefore the solution is given by

$$\boxed{\ln(y^2+a) = \frac{1}{3}\ln(1+x^3) + \ln(a^2+a)}$$

Other correct solutions include

$$y^2 + a = (a^2+a)(1+x^3)^{1/3} \text{ and } y = \sqrt{(a^2+a)(1+x^3)^{1/3} - a}.$$

2. (10 points) Find the general solution to

$$(e^{kx} - \frac{y}{x})dx - dy = 0,$$

on the interval $(0, \infty)$ where $k \neq 0$ is a non-zero constant. Give your answer in explicit form.

This is a first order linear differential equation.

$$y' + \frac{y}{x} = e^{kx}.$$

Th integrating factor is

$$I(x) = e^{\left(\int \frac{dx}{x}\right)} = e^{\ln(x)} = x.$$

Multiplying both sides of the equation by x and applying the product rule of differentiation gives

$$[xy]' = xe^{kx}.$$

Integrating

$$xy = \int xe^{kx}dx = x\,\frac{e^{kx}}{k} - \int \frac{e^{kx}}{k}dx = x\frac{e^{kx}}{k} - \frac{e^{kx}}{k^2} + C.$$

Therefore

$$\boxed{y = \frac{e^{kx}}{k} - \frac{e^{kx}}{k^2 x} + \frac{C}{x}}$$

3. (10 points) A tank initially contains 10 L of a salt solution. Pure water flows into the tank at a rate of 2 L/min, and the well-stirred mixture flows out at a rate of 3 L/min. After 5 min, the concentration of salt in the tank is 25 g/L. Find:

(a) The amount of salt in the tank initially.

(b) The volume of solution in the tank when the concentration of salt is 4 g/L.

Let $A(t)$ be the amount of salt at time t in g and $V(t)$ be the volume at time t in L.

$$V(t) = 10 - t$$

and

$$A' = \text{rate}_{\text{in}} - \text{rate}_{\text{out}} = 0 - 3\frac{A}{V} = -3\frac{A}{10-t}.$$

This is both a separable and first order linear differential equation. We solve it by separating variables

$$\frac{dA}{A} = 3\frac{-dt}{10-t} \implies \ln(A) = 3\ln(10-t) + C_1 \implies A = C\ln(10-t)^3.$$

To find C we set $t = 5$ and consider the concentration. Observe first that the concentration at time t equals

$$\frac{A(t)}{V(t)} = C(10-t)^2.$$

So

$$25 = C(10-5)^2 \implies C = 1.$$

Therefore

(a) $A(0) = (10-0)^3 = 1000$g.

(b) Note that the concentration is the square of the volume. Therefore, when the concentration is 4g/L, the volume is $\sqrt{4}L = 2$L.

4. (10 points) Determine all values of the constant a for which the following system has **i)** no solution, **ii)** an infinite number of solutions, **iii)** a unique solution.

$$\begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 2x_1 + 5x_2 - 4x_3 &= 10 \\ 2x_1 + 3x_2 + ax_3 &= 0 \end{aligned}$$

We row-reduce the augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 & | & 3 \\ 2 & 5 & -4 & | & 10 \\ 2 & 3 & a & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & | & 3 \\ 0 & 3 & -2 & | & 4 \\ 0 & 1 & a+2 & | & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & | & 3 \\ 0 & 1 & a+2 & | & -6 \\ 0 & 3 & -2 & | & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & | & 3 \\ 0 & 1 & a-2 & | & -6 \\ 0 & 0 & -3a-8 & | & 22 \end{bmatrix}.$$

i) When $a = -8/3$, the system is inconsistent (the last row implies $0\,x_1 + 0\,x_2 + 0\,x_3 = 22$).

iii) When $a \neq -8/3$, the system has a unique solution (by back substitution).

ii) So there is no value of a for which the system has infinitely many solutions.

Alternatively can use the rank of the augmented matrix, which is always equal to 3 (the "right side" gives three non-zero rows).

iii) When $a \neq 8/3$, the rank of the matrix is also 3 and so the system has a unique solution.

i) When $a = 8/3$, the rank of the matrix is $2 < 3$ and so the system has no solution.

ii) There is no value of a for which the rank of the augmented matrix is strictly smaller than 3 and so no value gives infinitely many solutions.

5. (10 points)

(a) Determine the inverse of the following matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}.$$

Row reduction gives

$$A^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

It is always a good idea to check that the A^{-1} you calculated indeed satisfies, say, $AA^{-1} = I_3$.

(b) Let

$$B = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 0 & -2 \\ -1 & 3 & 2 \end{bmatrix}.$$

Find the 3×3 matrix X that satisfies the matrix equation $XA + B = 0$, where 0 is the 3×3 zero matrix.

$$XA+B = 0 \implies XA = -B \implies XAA{-}1 = -BA^{-1} \implies X = -BA^{-1} = \begin{bmatrix} -1 & -8 & -6 \\ 2 & 0 & 5 \\ -2 & -2 & -2 \end{bmatrix}.$$

(c) What is the rank of the matrix A? Justify your answer briefly.

The rank of A is 3 because the matrix is invertible.