MTH 165: Linear Algebra with Differential Equations

2nd Midterm ANSWERS

April 4, 2015

1. (10 points)

(a) Let

$$A = \begin{bmatrix} -1 & 3 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Find the determinant of the matrix $C = ABA^2B^T$.

By determinant properties, $\det(C) = \det(A)^3 \cdot \det(B)^2$, so it suffices to find $\det(A)$ and $\det(B)$. For $\det(A)$ we expand along the bottom row, giving

$$\det(A) = \begin{vmatrix} -1 & 3 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & 3 \\ -1 & 1 \end{vmatrix} = -1.$$

For $\det(B)$ we expand along the top row, giving

$$\det(B) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 2.$$

Then $\det(C) = (-1)^3 \cdot 2^2 = -4$.

(b) Use the Wronskian to determine whether the functions $f_1(x) = \cos(x)$, $f_2(x) = \sin(x)$, and $f_3(x) = x$ are linearly independent.

The Wronskian is

$$W = \begin{vmatrix} \cos(x) & \sin(x) & x \\ -\sin(x) & \cos(x) & 1 \\ -\cos(x) & -\sin(x) & 0 \end{vmatrix} = x \cdot \begin{vmatrix} -\sin(x) & \cos(x) \\ -\cos(x) & -\sin(x) \end{vmatrix} - 1 \cdot \begin{vmatrix} \cos(x) & \sin(x) \\ -\cos(x) & -\sin(x) \end{vmatrix} = x.$$

This is not the zero function, so the functions are linearly independent. It is also possible to plug in a particular value of x, such as $x = \pi$, and then evaluate the determinant (obtaining a nonzero value) to conclude that the Wronskian is not identically zero.

2. (10 points) Determine whether each given set S is a subspace of the given vector space V. If so, give a proof; if not, explain why not.

(a) $V = \mathbb{R}^3$, and $S = \{(x, y, z) \in V \mid \ x^2 + y^2 + z^2 = 1\}$.

This <u>is not</u> a subspace as the zero vector $(0, 0, 0)$ does not lie in $S : 0^2 + 0^2 + 0^2 = 0 \neq 1$.

(b) $V = M_2(\mathbb{R})$, the set of 2×2 matrices, and $S = \{A \in V \mid \ \det(A) = 0\}$.

This <u>is not</u> a subspace as it is not closed under addition: both $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ belong to S, but their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ does not.

(c) $V = P_2(\mathbb{R})$, the set of polynomials of degree ≤ 2, and $S = \{f \in V \mid f(2) = 2f(1)\}$.

This <u>is</u> a subspace.

1. The zero polynomial $O(x) = 0$ for all x belongs in S: $O(2) = 0 = 2 \cdot 0 = 2O(1)$.
2. S is closed under addition: let f and g be two polynomials in S, their sum $(f + g)$ also belongs to S because

$$(f + g)(2) \stackrel{\text{def}}{=} f(2) + g(2) \stackrel{f,g \in S}{=} 2f(1) + 2g(1) = 2(f(1) + g(1)) \stackrel{\text{def}}{=} 2(f + g)(1).$$

3. S is closed under scalar multiplication: let f be a polynomial in S and λ a real number, their scalar product (λf) also belongs to S because

$$(\lambda f)(2) \stackrel{\text{def}}{=} \lambda f(2) \stackrel{f \in S}{=} \lambda(2f(1)) = 2(\lambda f(1)) \stackrel{\text{def}}{=} 2(\lambda f)(1).$$

3. (10 points) Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

(a) Do these vectors span $\mathbb{R}^3$? Explain why or why not.

The vectors span $\mathbb{R}^3$ if and only if the rank of the matrix whose columns are these vectors, or whose rows are these vectors, is 3. We will investigate the matrix whose columns are these vectors, since it will be helpful for part (b) as well:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 3 & 0 \\ 1 & 2 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Are these vectors linearly independent? If so, justify why; if not, find an explicit linear dependence between them.

No. Four vectors in a 3-dimensional space cannot be linearly independent.

Linear dependence of matrix columns is preserved in row-reduction, so any linear dependence among the columns in the row-reduced matrix corresponds to a linear dependence among the original vectors. Looking at the columns in the row-reduced matrix gives

$$2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_3 \ .$$

4. (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 3 \\ 2 & 4 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the row space of A.

We begin by bringing the matrix to row-echelon form.

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A basis for the row space of A is the set of non-zero rows of any row-echelon form. Therefore

$$\boxed{\text{basis for the row space of } A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.}$$

(b) Find a basis for the column space of A.

A basis for the column space of A consists of the columns of A that correspond to columns of a row-echelon form with no leading 1s.

In this case these are the first, second, and fourth columns. Therefore

$$\boxed{\text{basis for the column space of } A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}.}$$

5. (10 points) Answer the following about a 6×17 matrix A (that is, a matrix with 6 rows and 17 columns) such that $\text{rank}(A) = 6$.

(a) rowspace(A) is a $\underline{6}$-dimensional subspace of $\mathbb{R}^d$ with $d = \underline{17}$

Each row has 17 components, so the row space lives in $\mathbb{R}^{17}$, and the dimension of the row space is always equal to the rank of the matrix.

(b) colspace(A) is a $\underline{6}$-dimensional subspace of $\mathbb{R}^d$ with $d = \underline{6}$

Each column has 6 components, so the column space lives in $\mathbb{R}^6$, and the dimension of the column space is always equal to the rank of the matrix.

(c) nullspace(A) is a $\underline{11}$-dimensional subspace of $\mathbb{R}^d$ with $d = \underline{17}$

In order for the product $A\mathbf{v}$ to make sense, $\mathbf{v}$ must have 17 components, so null(A) lives in $\mathbb{R}^{17}$. By the rank-nullity theorem we know that rank(A)+dim(null(A))$= 17$, so dim(null(A))$= 11$.

(d) Are the rows of A linearly independent? Explain why or why not.

Since the dimension of the row space (6) is EQUAL to the number of rows, the rows must be linearly independent.

(e) Are the columns of A linearly independent? Explain why or why not.

Since the dimension of the column space (6) is LESS than the number of columns (17), the columns must be linearly dependent, in other words NOT linearly independent.