Math 165: Linear Algebra w/ Diff. Equations

Final (Morning Slot)

August 4th, 2022

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

- The exam will be 135 minutes long. You will get extra time in the end to upload the exam to Gradescope.
- Part A is worth 75 points. Part B is worth 105 points. Every question is labelled with one of these parts.
- There are 12 pages.
- A formula sheet is provided. If you find yourself in need of a formula that is not provided, feel free to ask.
- No calculators, phones, electronic devices, books, notes are allowed during the exam. The only materials you are allowed to use are are pen/pencil and paper. In particular, you are NOT allowed to take the exam on a tablet.
- You are allowed to use a phone or tablet to take photographs of your answer sheet once the exam is over. If you finish early, you must take permission before taking photographs. Once you start taking photographs, you are not allowed to write.
- **Show all work and justify all answers as much as possible.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

QUESTION	VALUE	SCORE
1	0	
2	20	
3	25	
4	30	
5	15	
6	30	
7	25	
8	10	
9	25	
TOTAL	180	

Formulas and Notation

- Power rule for integration

$$\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{if } n \neq -1 \\ \ln x + C & \text{if } n = -1 \end{cases}.$$

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- $$\int \frac{dx}{1+x^2} = \arctan x.$$

- $P_n(\mathbb{R})$ is the space of polynomials of degree $\leqslant n$. We have $\dim P_n(\mathbb{R}) = n+1$.
- $V_d(I)$ is the space of vector functions $\vec{x} : I \to \mathbb{R}^d$. This space is not finite dimensional.
- $C^r(I)$ is the space of functions $f : I \to \mathbb{R}$ that are r-times continuously differentiably.
- For $f_1, f_2, f_3 \in C^2(I)$, the Wronskian is given by

$$W[f_1, f_2, f_3](x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}.$$

- For $\vec{x}_1, \vec{x}_2, \vec{x}_3 \in V_3(I)$, the Wronskian is given by

$$W[\vec{x}_1, \vec{x}_2, \vec{x}_3](t) = \begin{vmatrix} \vec{x}_1(t) & \vec{x}_2(t) & \vec{x}_3(t) \end{vmatrix}.$$

1. (0 points) Copy the following honesty pledge on to your answer sheet. Remember to sign and date it.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

Part A

2. (20 points)

Solve the following differential equations / initial value problems.

(a)

$$\frac{dy}{dx} + 2xy = 2x^3.$$

(b)

$$\frac{dy}{dx} = 1 + x + y^2 + xy^2, \qquad y(0) = 0.$$

(c)

$$\frac{dy}{dx} + \frac{y}{x} = 4, \qquad y(1) = 2.$$

(d)

$$\left(\frac{dy}{dx}\right)^5 - x^{10}y = 0.$$

Part A

3. (25 points) Consider

$$A = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$$

(a) Compute $\det A$. Why is A invertible?

(b) Using elementary row operations, compute A^{-1}.

(c) Compute $\det A^{-3}$.

Part A

4. (30 points) Consider the matrix

$$\begin{bmatrix} 4 & 3 & 2 & -1 \\ 5 & 4 & 3 & -1 \\ -2 & -2 & -1 & 2 \\ 11 & 6 & 4 & 1 \end{bmatrix}.$$

(a) Find a basis for the row-space, and hence compute the dimension of the row-space.

(b) Find a basis for the column-space, and hence compute the dimension of the column-space.

(c) Find a basis for the null-space, and hence compute the dimension of the null-space.

(d) Do your answers in the previous parts match your expectation? Explain.

Part B

5. (15 points)

Determine whether the set S is linearly independent in the vector space V for the following.

(a) $V = P_{10}(\mathbb{R})$, $S = \{1, x+1, (x+1)^{10}\}$.

(b) $V = V_2(0,1)$, $S = \left\{ \begin{bmatrix} t^3 \\ t^2 \end{bmatrix}, \begin{bmatrix} t^2 \\ t^3 \end{bmatrix} \right\}$.

(c) $V = V_2(-\infty, \infty)$, $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \cos^2 t \\ \sin^2 t \\ 1 \end{bmatrix}, \begin{bmatrix} \sin^2 t \\ \cos^2 t \\ -1 \end{bmatrix} \right\}$

Part B

6. (30 points)

Compute the eigenvalues and eigenspaces of the following matrix.

$$\begin{bmatrix} 0 & 2 & -2 \\ -2 & 4 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

Part B

7. (25 points)

Let $P(D) = D^2 + 4D + 4$ and $Q(D) = D + 2$ be two polynomial differential operators.

(a) Show that $Q(D)$ annihilates e^{-2x}.

(b) Solve the differential equation $P(D)Q(D)y = 0$.

(c) Using your solution to the previous part, find the general solution to the differential equation $P(D)y = 2e^{-2x}$.

Part B

8. (10 points)

(a) Find a first order linear system of differential equations such that solving it is equivalent to solving the differential equation in 7(b).

(b) Rewrite this as a vector differential equation.

9. (25 points)

Consider

$$A = \begin{pmatrix} 6 & -5 \\ 5 & 6 \end{pmatrix}$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Using this, solve the initial value problem

$$\frac{d\vec{x}}{dt} = A\vec{x}, \qquad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$