# MATH 165 ( SUMMER '22, SESH B2)

## FINAL EXAM SOLUTIONS

(MORNING SLOT)

**2. (20 points)**

Solve the following differential equations / initial value problems.

(a)

$$\frac{dy}{dx} + 2xy = 2x^3.$$

INTEGRATING FACTOR $= e^{\int p(x)\,dx}$, $p(x) = 2x$

$= e^{\int 2x\,dx} = e^{x^2}$

$\therefore \frac{d}{dx}\left(e^{x^2} y\right) = e^{x^2}\frac{dy}{dx} + 2x e^{x^2} y = 2 e^{x^2} x^3$

$$\Rightarrow \quad e^{x^2} y = \int 2x^3 e^{x^2}\, dx$$

$$u = x^2 \quad \Rightarrow \quad du = 2x\,dx$$

$$\therefore \quad e^{x^2} y = \int x^2 e^{x^2} (2x\,dx) = \int u e^u\, du$$

$$= u e^u - \int e^u\, du$$

$$= u e^u - e^u + C$$

$$= x^2 e^{x^2} - e^{x^2} + C$$

$$\Rightarrow \boxed{y = x^2 - 1 + C e^{-x^2}}$$

(b)

$$\frac{dy}{dx} = 1 + x + y^2 + xy^2, \qquad y(0) = 0.$$

$$1 + x + y^2 + xy^2 = (1+x)(1+y^2)$$

$$\Rightarrow \quad \frac{dy}{1+y^2} = (1+x)\,dx$$

$$\Rightarrow \quad \int_0^y \frac{dy}{1+y^2} = \int_0^x (1+x)\,dx \qquad (\because y(0)=0)$$

$$\Rightarrow \text{Arctan}\, y = x + x^2/2 \Rightarrow y = \tan\left(x + x^2/2\right).$$

(c)

$$\frac{dy}{dx} + \frac{y}{x} = 4, \qquad y(1) = 2.$$

$$I.F. = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

$$\therefore \frac{d}{dx}(xy) = x\frac{dy}{dx} + y = 4x$$

$$\Rightarrow \int_1^x \frac{d}{dx}(xy)\,dx = \int_1^x 4x\,dx = 2x^2 - 2$$

$$\Rightarrow xy - (1)(2) = 2x^2 - 2 \qquad (\because y(1) = 2)$$

$$\Rightarrow y = 2x.$$

(d)

$$\left(\frac{dy}{dx}\right)^5 - x^{10}y = 0.$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(x^{10}\,y\right)^{1/5} = x^2\,y^{1/5}$$

$$\Rightarrow \quad y^{-1/5}\,dy = x^2\,dx$$

$$\Rightarrow \quad \frac{y^{4/5}}{4/5} = \frac{x^3}{3} + C \Rightarrow y = \left(\frac{4x^3}{15} + C'\right)^{5/4}$$

**3. (25 points)** Consider

$$A = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$$

(a) Compute $\det A$. Why is $A$ invertible?

(b) Using elementary row operations, compute $A^{-1}$.

(c) Compute $\det A^{-3}$.

(c) $\det A^{-3} = (\det A)^{-3}$

$= 1^{-3} = 1$

(a) $\det A = 2 \begin{vmatrix} -2 & 2 \\ 6 & -7 \end{vmatrix} - 3 \begin{vmatrix} -1 & 2 \\ 4 & -7 \end{vmatrix} + (-3) \begin{vmatrix} -1 & -2 \\ 4 & 6 \end{vmatrix}$

$= 2(2) - 3(-1) + (-3)(2) = 1 \neq 0$

$\Leftrightarrow$ A IS INVERTIBLE.

(b) $\left[\begin{array}{ccc|ccc} 2 & 3 & -3 & 1 & 0 & 0 \\ -1 & -2 & 2 & 0 & 1 & 0 \\ 4 & 6 & -7 & 0 & 0 & 1 \end{array}\right]$

$$\Big\downarrow \text{EROs}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

**4. (30 points)** Consider the matrix

$$\begin{bmatrix} 4 & 3 & 2 & -1 \\ 5 & 4 & 3 & -1 \\ -2 & -2 & -1 & 2 \\ 11 & 6 & 4 & 1 \end{bmatrix}.$$

SEE SOLNS TO SAMPLE MIDTERM 2.

**5. (15 points)**

Determine whether the set $S$ is linearly independent in the vector space $V$ for the following.

(a) $V = P_{10}(\mathbb{R})$, $S = \{1, x+1, (x+1)^{10}\}$.

$$W = \begin{vmatrix} 1 & x+1 & (x+1)^{10} \\ 0 & 1 & 10(x+1)^9 \\ 0 & 0 & 90(x+1)^8 \end{vmatrix} = 90(x+1)^8 \neq 0$$

(e.g. if $x=0$).

$\therefore$ L.I.

(b) $V = V_2(0,1)$, $S = \left\{ \begin{bmatrix} t^3 \\ t^2 \end{bmatrix}, \begin{bmatrix} t^2 \\ t^3 \end{bmatrix} \right\}$.

$$W = \begin{vmatrix} t^3 & t^2 \\ t^2 & t^3 \end{vmatrix} = t^6 - t^4 \neq 0 \quad (\text{e.g. if } t = 1/2)$$

$\therefore$ L.I.

(c) $V = V_2(-\infty, \infty)$, $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \cos^2 t \\ \sin^2 t \\ 1 \end{bmatrix}, \begin{bmatrix} \sin^2 t \\ \cos^2 t \\ -1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos^2 t \\ \sin^2 t \\ 1 \end{bmatrix} + \begin{bmatrix} \sin^2 t \\ \cos^2 t \\ -1 \end{bmatrix}$$

$\therefore$ L.D.

**6. (30 points)**

Compute the eigenvalues and eigenspaces of the following matrix.

$$\begin{bmatrix} 0 & 2 & -2 \\ -2 & 4 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

SEE SOLN. TO SAMPLE FINAL.

**7. (25 points)**

Let $P(D) = D^2 + 4D + 4$ and $Q(D) = D + 2$ be two polynomial differential operators.

(a) Show that $Q(D)$ annihilates $e^{-2x}$.

(b) Solve the differential equation $P(D)Q(D)y = 0$.

(c) Using your solution to the previous part, find the general solution to the differential equation $P(D)y = 2e^{-2x}$. (A)

(a) $Q(D)e^{-2x} = (D+2)e^{-2x} = D(e^{-2x}) + 2e^{-2x}$

$= -2e^{-2x} + 2e^{-2x} = 0$

(b) $(D^2+4D+4)(D+2)\,y=0$

AUXILIARY EQN $\quad (\lambda^2+4D+4)(\lambda+2)=0$

$\Rightarrow (\lambda+2)^3=0$

$\Rightarrow \lambda=-2 \quad$ (w/ MULT. 3)

$$\therefore\ y=\underbrace{c_1e^{-2x}+c_2xe^{-2x}}_{P(D),\ y_c}+\underbrace{c_3x^2e^{-2x}}_{Q(D),\ y_p}$$

(c) PLUGGING $y_p$ INTO (A):

$$P(D)\,y_p = 2e^{-2x}$$

$$(D^2+4D+4)(c_3 x^2 e^{-2x}) = 2e^{-2x}$$

$$\Rightarrow \quad 2c_3 e^{-2x} = 2e^{-2x}$$

$$\Rightarrow \quad c_3 = 1$$

$$\therefore \quad y = y_c + y_p = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 e^{-2x}$$

**8. (10 points)**

(a) Find a first order linear system of differential equations such that solving it is equivalent to solving the differential equation in 7(b).

(b) Rewrite this as a vector differential equation.

$$\underline{(D^2+4D+4)}(D+2)\,y=0 \rightsquigarrow \frac{d^3y}{dt^3}+6\frac{d^2y}{dt^2}+12\frac{dy}{dt}+8y=0$$

$$x_1=y, \quad x_2=\frac{dy}{dt}, \quad x_3=\frac{d^2y}{dt^2}$$

$\Rightarrow$ SYSTEM: $\quad dx_1/dt = x_2, \quad dx_2/dt = x_3$

$$dx_3/dt = -8x_1 - 12x_2 - 6x_3$$

$$\Rightarrow \quad \frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{bmatrix} \vec{x}$$

Consider

$$A = \begin{pmatrix} 6 & -5 \\ 5 & 6 \end{pmatrix}$$

(a) Find the eigenvalues and eigenvectors of $A$.

(b) Using this, solve the initial value problem

$$\frac{d\vec{x}}{dt} = A\vec{x}, \qquad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$p(\lambda) = \begin{vmatrix} 6-\lambda & -5 \\ 5 & 6-\lambda \end{vmatrix} = (6-\lambda)^2 + 25$$

$$\Rightarrow \text{E.V.} = \lambda = 6 \pm 5i$$

FOR $\lambda = 6 - 5i$

$$A - \lambda I = \begin{bmatrix} 5i & -5 \\ 5 & 5i \end{bmatrix}$$

$$\Big\downarrow \text{ERO}$$

$$\begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}$$

$\therefore\ i v_1 - v_2 = 0$

$\therefore\ v_1 = t,\quad v_2 = it \implies \begin{bmatrix} 1 \\ i \end{bmatrix}$ is an e.v.

$\therefore$ $6+5i$, $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ IS THE OTHER EIGENPAIR.

(b) $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

WHERE

$$\vec{x}_1 = e^{(6+5i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\vec{x}_2 = e^{(6-5i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\therefore \quad c_1 x_1(0) + c_2 x_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad c_1 + c_2 = 0$$
$$-ic_1 + ic_2 = 1$$

$$\Rightarrow \quad c_1 = \frac{-1}{2i}, \quad c_2 = \frac{1}{2i}$$

$$\therefore \vec{x} = -\frac{1}{2i} x_1(t) + \frac{1}{2i} x_2(t)$$

$$= \begin{bmatrix} -e^{6t} \sin 5t \\ e^{6t} \cos 5t \end{bmatrix}$$