MATH 165 (SUMMER '22, SESH B2)

FINAL EXAM SOLUTIONS

(EVENING SLOT)

2.

(a)

$$\frac{dy}{dx} + 3x^2 y = 3x^5.$$

INTEGRATING FACTOR $= e^{\int p(x)\,dx}$, $p(x) = 3x^2$

$= e^{x^3}$

$\therefore \quad \frac{d}{dx}\left(e^{x^3} y\right) = e^{x^3}\frac{dy}{dx} + 3x^2 e^{x^3} y = 3e^{x^3} x^5$

$$\Rightarrow \quad e^{x^3} y = \int 3 x^5 e^{x^3} \, dx$$

$$u = x^3 \quad \Rightarrow \quad du = 3x^2 \, dx$$

$$\therefore \quad e^{x^3} y = \int x^3 e^{x^3} (3x^2 \, dx) = \int u e^u \, du$$

$$= u e^u - \int e^u \, du$$

$$= u e^u - e^u + C$$

$$= x^3 e^{x^3} - e^{x^3} + C$$

$$\Rightarrow \boxed{y = x^3 - 1 + C e^{-x^3}}$$

(b)

$$\frac{dy}{dx} = x + x^2 + xy^2 + x^2y^2, \qquad y(0) = 0.$$

$$x + x^2 + xy^2 + x^2y^2 = (x+x^2)(1+y^2)$$

$$\Rightarrow \quad \frac{dy}{1+y^2} = (x+x^2)\,dx$$

$$\Rightarrow \quad \int_0^y \frac{dy}{1+y^2} = \int_0^x (x+x^2)\,dx \qquad (\because y(0)=0)$$

$$\Rightarrow \text{Arctan}\, y = \frac{x^2}{2} + \frac{x^3}{3} \quad \Rightarrow \quad y = \tan\left(x^2/2 + x^3/2\right)$$

(c)

$$\frac{dy}{dx} + \frac{y}{x} = 4, \qquad y(1) = 2.$$

$$I.F. = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

$$\therefore \frac{d}{dx}(xy) = x\frac{dy}{dx} + y = 4x$$

$$\Rightarrow \int_1^x \frac{d}{dx}(xy)\,dx = \int_1^x 4x\,dx = 2x^2 - 2$$

$$\Rightarrow xy - (1)(2) = 2x^2 - 2 \qquad (\because y(1)=2)$$

$$\Rightarrow y = 2x.$$

(d)

$$\left(\frac{dy}{dx}\right)^7 - x^7 y = 0.$$

$$\Rightarrow \quad \frac{dy}{dx} = (x^7 y)^{1/7} = x\, y^{1/7}$$

$$\Rightarrow \quad y^{-1/7}\, dy = x\, dx$$

$$\Rightarrow \quad \frac{y^{6/7}}{6/7} = \frac{x^2}{2} + C \Rightarrow y = \left(\frac{3x^3}{7} + C'\right)^{7/6}$$

3. (25 points) Consider

$$A = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$$

(a) Compute $\det A$. Why is A invertible?

(b) Using elementary row operations, compute A^{-1}.

(c) Compute $\det A^{-1} + \det A$.

(c) $\det A^{-1} + \det A$

$= \dfrac{1}{\det A} + \det A$

$= \dfrac{1}{1} + 1 = 2$

(a) $\det A = 2 \begin{vmatrix} -2 & 2 \\ 6 & -7 \end{vmatrix} - 3 \begin{vmatrix} -1 & 2 \\ 4 & -7 \end{vmatrix} + (-3) \begin{vmatrix} -1 & -2 \\ 4 & 6 \end{vmatrix}$

$= 2(2) - 3(-1) + (-3)(2) = 1 \neq 0$

$\Rightarrow$ A IS INVERTIBLE.

(b) $$\left[\begin{array}{ccc|ccc} 2 & 3 & -3 & 1 & 0 & 0 \\ -1 & -2 & 2 & 0 & 1 & 0 \\ 4 & 6 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \text{EROs}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

4. (30 points) Consider the matrix

$$\begin{bmatrix} 4 & 3 & 2 & -1 \\ 5 & 4 & 3 & -1 \\ -2 & -2 & -1 & 2 \\ 11 & 6 & 4 & 1 \end{bmatrix}.$$

SEE SOLNS TO SAMPLE MIDTERM 2.

5. (15 points)

Determine whether the set S is linearly independent in the vector space V for the following.

(a) $V = P_{10}(\mathbb{R})$, $S = \{1, 1-x, (1-x)^{10}\}$.

$$W = \begin{vmatrix} 1 & 1-x & (1-x)^{10} \\ 0 & -1 & -10(1-x)^{9} \\ 0 & 0 & 90(1-x)^{8} \end{vmatrix} = -90(1-x)^{8} \neq 0$$

(e.g. if $x=0$).

$\therefore$ L.I.

(b) $V = V_2(0,1)$, $S = \left\{ \begin{bmatrix} t \\ t^4 \end{bmatrix}, \begin{bmatrix} t^4 \\ t \end{bmatrix} \right\}$.

$$W = \begin{vmatrix} t & t^4 \\ t^4 & t \end{vmatrix} = t^2 - t^8 \neq 0 \quad (\text{e.g. if } t = 1/2)$$

$\therefore$ L.I.

(c) $V = V_2(-\infty, \infty)$, $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} \cos^2 t \\ \sin^2 t \\ 1 \end{bmatrix}, \begin{bmatrix} \sin^2 t \\ \cos^2 t \\ 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \cos^2 t \\ \sin^2 t \\ 1 \end{bmatrix} + \begin{bmatrix} \sin^2 t \\ \cos^2 t \\ 1 \end{bmatrix}$$

$\therefore$ L.D.

6. (30 points)

Compute the eigenvalues and eigenspaces of the following matrix.

$$\begin{bmatrix} 0 & 2 & -2 \\ -2 & 4 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

SEE SOLN. TO SAMPLE FINAL.

7. (25 points)

Let $P(D) = D^2 + 3D + 2$ and $Q(D) = D + 2$ be two polynomial differential operators.

(a) Show that $Q(D)$ annihilates e^{-2x}.

(b) Solve the differential equation $P(D)Q(D)y = 0$.

(c) Using your solution to the previous part, find the general solution to the differential equation $P(D)y = 2e^{-2x}$. – B

(a) $Q(D)e^{-2x} = (D+2)e^{-2x} = D(e^{-2x}) + 2e^{-2x}$

$= -2e^{-2x} + 2e^{-2x} = 0$

(b) $(D^2+3D+2)(D+2)\,y=0$

AUXILIARY EQN $(\lambda^2+3\lambda+2)(\lambda+2)=0$

$\Rightarrow (\lambda+2)^2(\lambda+1)$

$\Rightarrow \lambda=-2$ (w/ MULT. 2)

$\lambda=-1$ (w/ MULT. 1)

$$\therefore\; y = \underbrace{c_1 e^{-2x} + c_2 e^{-x}}_{P(D),\ y_c} + \underbrace{c_3 x e^{-2x}}_{Q(D),\ y_p}$$

(C) PLUGGING y_p INTO (A) :

$$P(D)\,y_p = 2e^{-2x}$$

$$(D^2+3D+2)(c_3 x e^{-2x}) = 2e^{-2x}$$

$$\Rightarrow c_3 e^{-2x} = 2e^{-2x}$$

$$\Rightarrow c_3 = 2$$

$$\therefore y = y_c + y_p = c_1 e^{-2x} + c_2 e^{-x} + 2x e^{-2x}$$

8. (10 points)

(a) Find a first order linear system of differential equations such that solving it is equivalent to solving the differential equation in 7(b).

(b) Rewrite this as a vector differential equation.

$$(D^2+3D+2)(D+2)\,y=0 \rightsquigarrow \frac{d^3y}{dt^3}+5\frac{d^2y}{dt^2}+8\frac{dy}{dt}+4y=0$$

$$x_1=y\ ,\quad x_2=\frac{dy}{dt}\ ,\quad x_3=\frac{d^2y}{dt^2}$$

$\Rightarrow$ SYSTEM: $\quad dx_1/dt=x_2\ ,\quad dx_2/dt=x_3$

$$dx_3/dt=-4x_1-8x_2-5x_3$$

$$\Rightarrow \quad \frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \vec{x}$$

9. (25 points)

Consider

$$A = \begin{pmatrix} 5 & -6 \\ 6 & 5 \end{pmatrix}$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Using this, solve the initial value problem

$$\frac{d\vec{x}}{dt} = A\vec{x}, \qquad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$p(\lambda) = \begin{vmatrix} 5-\lambda & -6 \\ 6 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 + 36$$

$$\Rightarrow \text{E.V.} = \lambda = 5 \pm 6i$$

FOR $\lambda = 5 - 6i$

$$A - \lambda I = \begin{bmatrix} 6i & -6 \\ 6 & 6i \end{bmatrix}$$

$$\Big\downarrow \text{ERO}$$

$$\begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}$$

$\therefore iv_1 - v_2 = 0$

$\therefore v_1 = t, \quad v_2 = it \quad \Rightarrow \begin{bmatrix} 1 \\ i \end{bmatrix}$ is an e.v.

$\therefore \quad 5+6i, \begin{bmatrix} 1 \\ -i \end{bmatrix}$ IS THE OTHER EIGENPAIR.

(b) $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

WHERE

$$\vec{x}_1 = e^{(5+6i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\vec{x}_2 = e^{(5-6i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\therefore \quad c_1 x_1(0) + c_2 x_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} c_1 + c_2 &= 0 \\ -ic_1 + ic_2 &= 1 \end{aligned}$$

$$\Rightarrow \quad c_1 = \frac{-1}{2i}, \quad c_2 = \frac{1}{2i}$$

$$\therefore \vec{x} = -\frac{1}{2i} x_1(t) + \frac{1}{2i} x_2(t)$$

$$= \begin{bmatrix} -e^{5t} \sin 6t \\ e^{5t} \cos 6t \end{bmatrix}$$