MATH 165 (SUMMER '22, SESH B2)

ANURAG SAHAY

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LECTURES:

9:00 AM – 11:15 AM (ET)

M, T, W, R

BREAK

~ 10-15 MIN

COURSE WEBPAGE

https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

COURSE INFORMATION

1. TEXTBOOK & REQUIREMENTS (WORKING WEBCAM!)

2. PREREQUISITES → 143, 162, 172 OR EQUIV.
→ 164 IS NOT A PREREQ.

3. (TENTATIVE) SCHEDULE & COURSE DESCRIPTION

* ALL LECTURES WILL BE RECORDED
* OFFICE HOURS WILL NOT BE RECORDED

COURSE INFORMATION

4. EXAMS

MIDTERM 1 : MONDAY, JULY 11th

MIDTERM 2 : MONDAY, JULY 25th

FINAL EXAM : THURSDAY, AUGUST 4th

PART A OF FINAL WILL REPLACE A MIDTERM IF THIS IMPROVES YOUR GRADE.

5. HOMEWORK (VIA WEBWORK)

* ~2 PER WEEK, LOGIN VIA BLACKBOARD

* WW 00 DUE TUESDAY (28th JUN), WW 01 DUE SATURDAY (2nd JULY)

6. GRADING

7. POLICIES (a) ACADEMIC HONESTY

(b) DISABILITY SUPPORT

(c) ZOOM

8. COURSE TA: PABLO BHOWMIK → ROLE TBD

OTHER NOTES

1. WEEKLY FEEDBACK FORM
2. WILL UPLOAD LECTURE NOTES & LECTURES
3. MINIMAL USE OF BLACKBOARD
   (ONLY GRADES, WEBWORK LOGIN, & EMAILS)
4. PLEASE KEEP YOUR VIDEOS ON, IF POSSIBLE!
5. DON'T FALL BACK (ONLY 6 WEEKS!)
6. WE WILL TAKE BASIC CALC FOR GRANTED.

§ 1.1 DIFFERENTIAL EQUATIONS EVERYWHERE

Q. WHAT IS A DIFFERENTIAL EQUATION?

$y=f(x)$

$y' = y$

$y=3e^x$

$y = Ce^x$

$C \in \mathbb{R}$

2nd (ORDER 2)

e.g. $\boxed{\frac{d^2y}{dx^2}} + x^2\frac{dy}{dx} + y^2 = 5\sin x$

1st

e.g. $\boxed{\frac{ds}{dt}} = e^{3t}(s-1)$

e.g. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ P.D.E.

ORDINARY D.E.

ONE INDEPENDENT VAR. x

" DEPEND. VAR. y

$$y = f(x)$$

Def.

1. (IN) DEPENDENT VARIABLE → IND. VAR. → x, t PARAMETER

→ DEP VAR. → y (DEPENDS OF x, t)

2. ODE vs. PDE

3. ORDER → LARGEST DERIVATIVE THAT APPEARS IN A DIFF EQN.

Q. WHY STUDY DEs ?

Ans :

e.g.1 MALTHUSIAN MODEL

$P :=$ POPULATION OF BACTERIA

$$\boxed{\frac{dP}{dt} \propto P} \longrightarrow \frac{dP}{dt} = kP$$

$k \to$ (PROPORTIONALITY CONSTANT)

$$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$\frac{dP}{dt} = kP$$

$$\Rightarrow \frac{d}{dt}\left(\ln P\right) = \frac{1}{P}\frac{dP}{dt} = k$$

$$\frac{d}{dt}(\ln P) = \left(\frac{dP}{dt}\right) \cdot \frac{d(\ln P)}{dP} = \frac{1}{P}\frac{dP}{dt}$$

$$\Rightarrow \frac{d}{dt}(\ln P) = k \Rightarrow \ln P = \int k\,dt = kt + c$$

$C = e^{c}$

$\ln P = kt + c$

$P(t) = e^{\ln P} = e^{(kt + c)} = Ce^{kt}$

$e^{kt} \cdot e^{c}$

$$\boxed{P(t) = Ce^{kt}}$$

GENERAL SOLN.

Def:

4. GENERAL SOLN. $\rightsquigarrow$ AN EXPRESSION w/ PARAMETERS (CONSTANTS OF INT.) SUCH THAT EVERY CHOICE OF PARAMS. GIVES A SOLN.

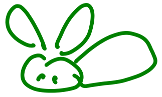
[EXHAUSTIVE]

5. PARTICULAR SOLN. $\rightsquigarrow$ A SOLUTION w/ PARAMS. FIXED.

e.g. $P(t) = Ce^{kt}$ (GEN SOLN. OF $\frac{dP}{dt} = kP$)

say $C = 2$: $P(t) = 2e^{kt}$

PARTICULAR SOLN.



e.g.2 LOGISTIC MODEL

* BIRTH RATE $\propto$ POPULATION
* DEATH RATE $\propto$ POPULATION^2

$$\frac{dP}{dt} = \text{BIRTH RATE} - \text{DEATH RATE}$$
$$= B_0 P - D_0 P^2$$

$B_0, D_0 \rightarrow$ CONSTANTS

$$\boxed{\frac{dP}{dt} = P(B_0 - D_0 P)}$$

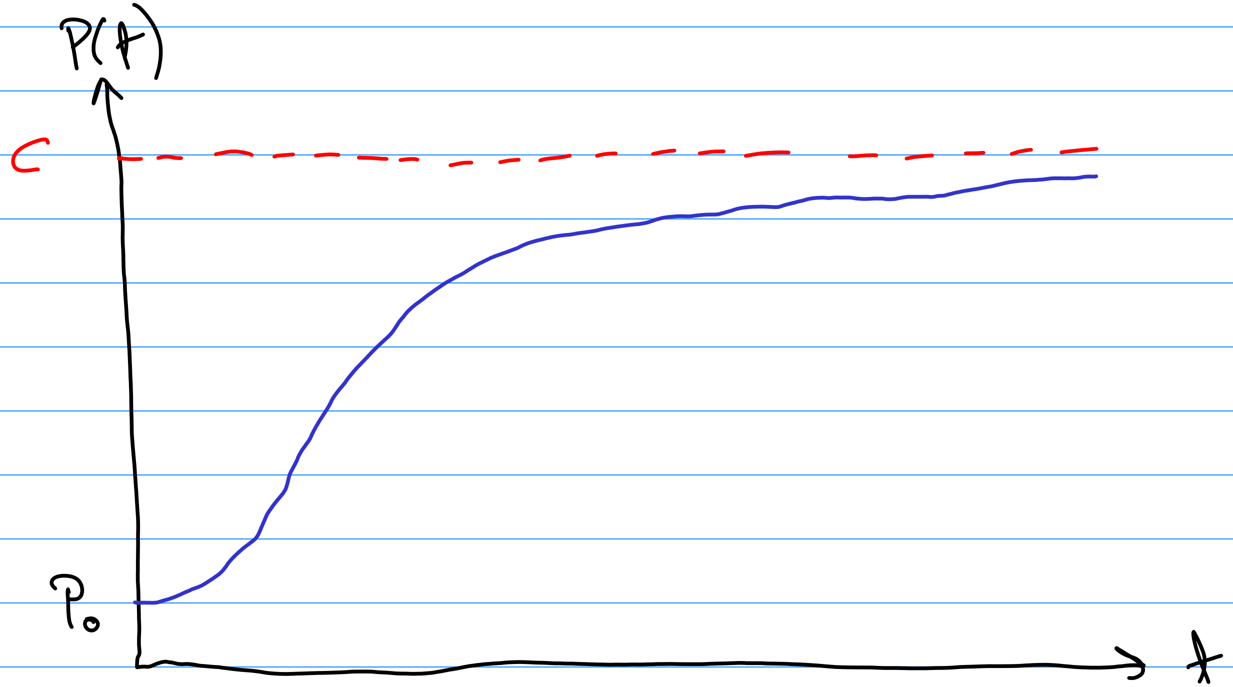
$\rightarrow$ LOGISTIC EQUATION

$k = B_0$

$$\frac{dP}{dt} = k\left(1 - \frac{P}{C}\right)P$$

$C = B_0 / D_0$

CARRYING CAPACITY

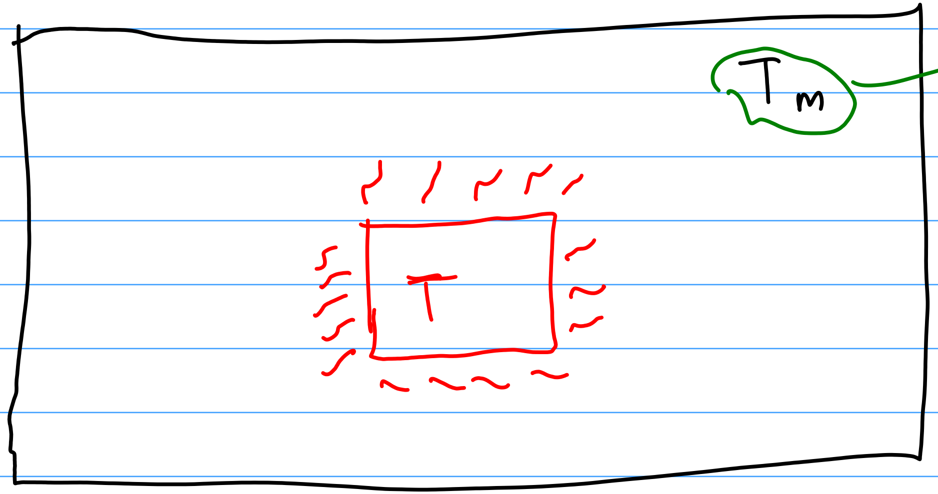


TYPICAL GROWTH OF LOGISTIC P(t)

BREAK TILL

10 : 10 AM

e.g. 3 NEWTON'S LAW OF COOLING



ROOM TEMP

$T - T_m$

RATE OF COOLING $\propto$ TEMPERATURE GRADIENT

$$\frac{dT}{dt} \propto (T - T_m)$$

$$\frac{dT}{dt} = -k(T - T_m) \qquad (k > 0)$$

T_m : CONSTANT

$$\left(\frac{1}{T - T_m}\right)\frac{dT}{dt} = -k \Rightarrow \frac{d}{dt}\left(\ln(T - T_m)\right) = -k$$

$$\frac{d}{dt}\left(\ln(T - T_m)\right)$$

$$\ln(T - T_m) = \int -k\,dt = -kt + c$$

$$T - T_m = e^{-kt + c}$$

$$T_0 - T_m = e^{c}$$

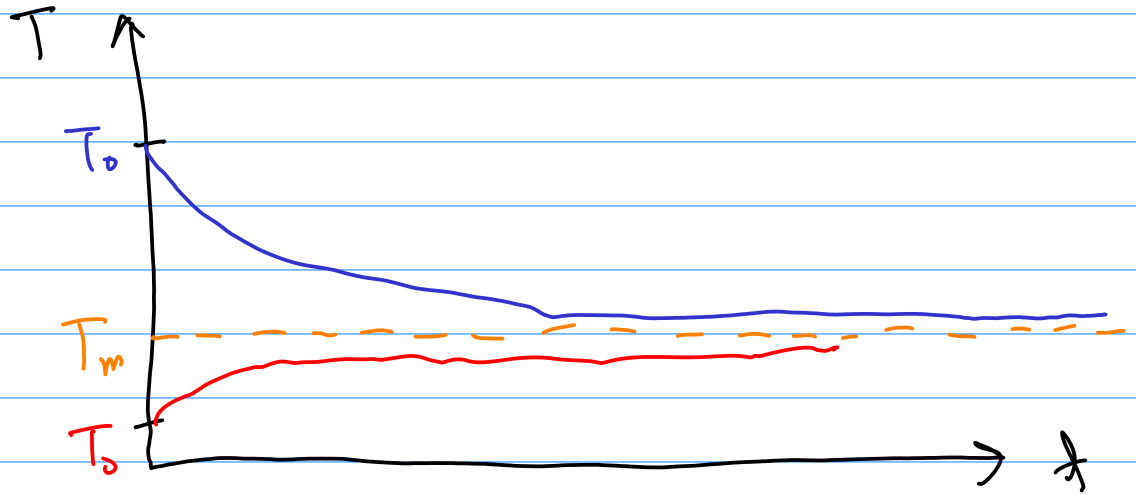
$t = 0$

$(T_0 \rightarrow$ TEMP. AT TIME $= 0)$

$$T - T_m = (T_0 - T_m)\, e^{-kt}$$

$$\boxed{T(t) = T_m + (T_0 - T_m)\, e^{-kt}}$$

ASSUMPTION : T_m IS CONSTANT



e.g. 4 NEWTON'S 2nd LAW OF MOTION

$$ma = F$$

F → FORCE
m → MASS
a → ACCELERATION

$$a = \frac{dv}{dt}, \quad v = \frac{dy}{dt}$$

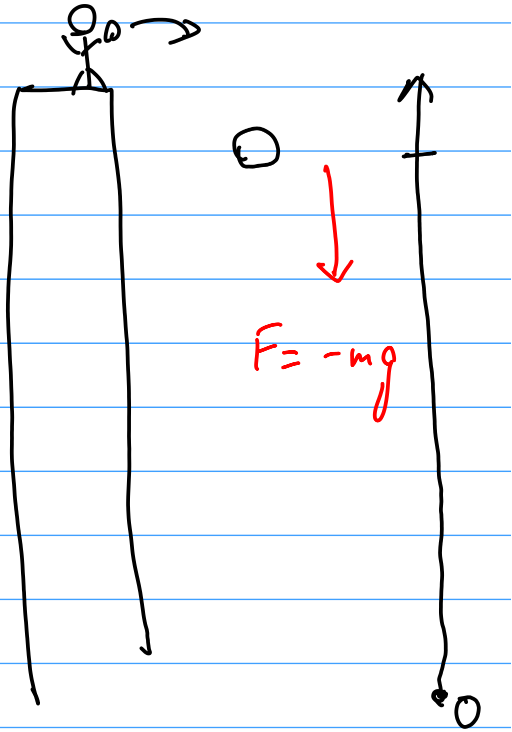
y → POSITION
v → VELOCITY

$$a = \frac{d^2y}{dt^2}$$

$$m\frac{d^2y}{dt^2} = F$$

→ VERTICAL MOTION UNDER GRAVITY

$F \equiv mg$ $\quad$ $g \rightsquigarrow$ UNIVERSAL CONSTANT



$$m\frac{d^2y}{dt^2} = -mg$$

$$\Rightarrow \frac{d^2y}{dt^2} = -g$$

$$\Rightarrow \frac{dy}{dt} = \int -g\,dt = -gt + c_1$$

$$v(t) = \frac{dy}{dt} = -gt + c_1$$

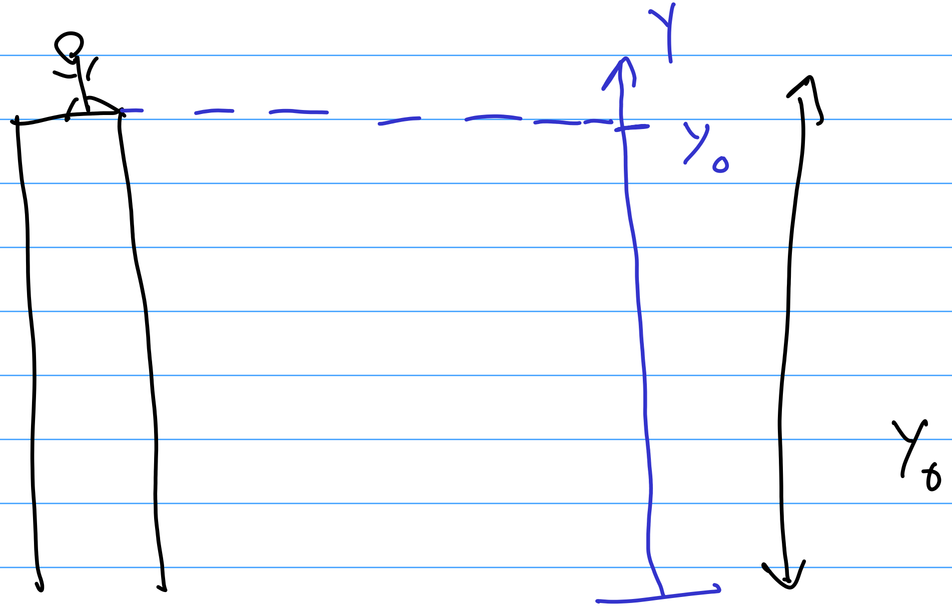
$$y(t) = \int (-gt + c_1)\,dt$$

$$y(t) = -\frac{gt^2}{2} + c_1 t + c_2$$

$$y_0 = y(0) = c_2$$

$$v_0 = v(0) = c_1$$

$$y(t) = -\frac{1}{2} g t^2 + v_0 t + y_0$$



Def:

6. INITIAL VALUE PROBLEM

DATA = DIFF EQN. + ENOUGH STARTING CONDITIONS

OTHER EXAMPLES (SEE BOOK)

* ORTHOGONAL TRAJECTORIES
* SPRING FORCE
* ONTOGENETIC GROWTH

§ 1.2 BASIC IDEAS AND TERMINOLOGY

(ORDINARY)

DIFF. EQN. OF ORDER n

$$G(x, y, y', \dots, y^{(n)}) = 0$$

LINEAR IF: G DEPENDS LINEARLY ON $y, y', y'', \dots, y^{(n)}$

yy' $\quad y^2 \quad \sin y$

Def.

7. LINEAR ODE.

$a_0(x) \not\equiv 0$

DEFINITION 1.2.1

A differential equation that can be written in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x),$$

where $a_0, a_1, \ldots, a_n$ and F are functions of x only, is called a **linear** differential equation of order n. Such a differential equation is linear in $y, y', y'', \ldots, y^{(n)}$.

8. NONLINEAR ODE

→ EVERYTHING ELSE

3rd

1. $\boxed{y'''} + e^{3x}y'' + x^3y' + (\cos x)y = \ln x$

a. ORDER

b. LINEAR/NONLINEAR

LINEAR

ORDER 2

3. $\boxed{y''} + \boxed{y^2} = 0$

QUADRATIC IN Y!

NONLINEAR

ORDER 1

a. ORDER

b. LINEAR/NON LINEAR

$$a_1(x)$$

2. $xy' - \dfrac{2}{1+x^2}y = 0 \rightarrow F(x)$

$a_0(x)$

LINEAR

ORDER 2

4. $y'' + x^4\boxed{\cos(y')} - xy = e^{x^2}$

$\rightarrow$ cos y' $\rightarrow$ NOT LINEAR !

EXTRA

$$\boxed{y\,y'} - x = 0$$

NONLINEAR

ORDER 1

1. ORDER ?
2. (NON) LINEAR ?

The general forms for first- and second-order linear differential equations are

A : $$a_0(x)\frac{dy}{dx} + a_1(x)y = F(x)$$ (1st ORDER LINEAR DE)

and

B: $$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x)$$ (2nd ORDER LINEAR DE)

respectively.

→ MALTHUSIAN MODEL (A) : $dP/dt = kP$

→ LOGISTIC MODEL [NEITHER] : $dP/dt = B_0 P - D_0 P^2$ → NON-LI

→ NEWTON'S LAW OF COOLING (A) : $dT/dt = -k(T - T_m)$

→ NEWTON'S 2nd LAW OF MOTION (B) : $m\frac{d^2y}{dt^2} = -mg$

Def (SOLUTION TO AN ODE)

DEFINITION 1.2.4

A function $y = f(x)$ that is (at least) n times differentiable on an interval I is called a **solution** to the differential equation (1.2.1) on I if the substitution $y = f(x)$, $y' = f'(x), \ldots, y^{(n)} = f^{(n)}(x)$ reduces the differential equation (1.2.1) to an identity valid for all x in I. In this case we say that $y = f(x)$ **satisfies** the differential equation.

$y' = y$ $\quad\quad$ $y = e^x$ $\quad\quad$ $y' = e^x$

$e^x = e^x$ (IDENTITY) $\quad\quad$ $\Rightarrow f(x) = e^x$ IS A SOLN. TO $y' = y$

$$y'' + y = 0 \quad - \quad \text{(I)}$$

$$\boxed{f(x) = \sin x} \longrightarrow \text{SOLN. TO (I)}$$

$$f'(x) = \cos x$$

$$f''(x) = \frac{d}{dx}(\cos x) = -\sin x$$

$y = f(x)$ IN (I)

$$(-\sin x) + (\sin x) = 0$$

Example 1.2.5 Verify that for all constants c_1 and c_2, $\boxed{y(x) = c_1e^{2x} + c_2e^{-2x}}$ is a solution to the linear differential equation $\boxed{y'' - 4y = 0}$ for x in the interval $(-\infty, \infty)$.

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(c_1 e^{2x} + c_2 e^{-2x}\right)$$

$$= c_1 \frac{d}{dx}\left(e^{2x}\right) + c_2 \frac{d}{dx}\left(e^{-2x}\right)$$

$$= c_1\left(2e^{2x}\right) + c_2\left(-2e^{-2x}\right)$$

$$= 2c_1 e^{2x} - 2c_2 e^{-2x}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(2c_1 e^{2x} - 2c_2 e^{-2x}\right)$$

$$= 4c_1 e^{2x} + 4c_2 e^{-2x}$$

PLUG IN $y'' - 4y =$

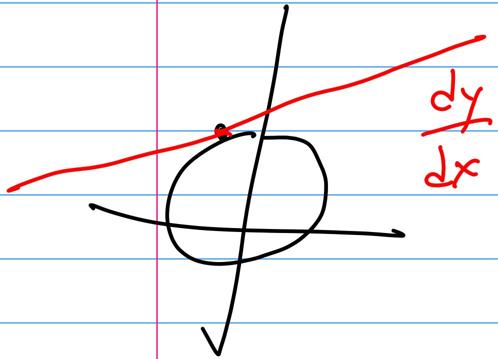
$$\left(4c_1 e^{2x} + 4c_2 e^{-2x}\right) - 4\left(c_1 e^{2x} + c_2 e^{-2x}\right)$$

$$= 0$$

$$\frac{d}{dx}(y^2) = \frac{dy}{dx} \cdot \frac{dy^2}{dy} = 2y\,y'$$

IMPLICIT DIFF.

$$x^2 + y^2 = 4 \Rightarrow \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(4) = 0$$



$\frac{dy}{dx}$ = SLOPE OF TANGENT

$$2x + 2y\,y' = 0$$

$$\frac{dy}{dx} = y' = \frac{-2x}{2y} = -\frac{x}{y}$$

Example 1.2.6 Verify that the relation $\boxed{x^2 + y^2 - 4 = 0}$ defines an implicit solution to the nonlinear differential equation

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}.}$$