MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
T – 9:00 PM – 10:00 PM (ET)
F – 3:00 PM - 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEBPAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1-9 ARE UPLOADED.

2. WW 04 - IS DUE TODAY (13th JULY) AT 11:00 PM ET
   WW 05 - IS DUE SATURDAY (16th JULY) AT 11:00 PM ET

3. MIDTERM HAS BEEN GRADED. → REGRADE REQUESTS ARE OPEN TILL FRIDAY

4. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE!

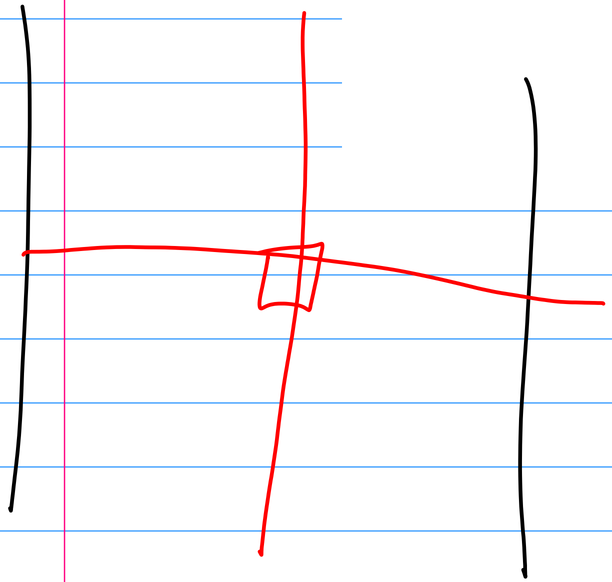
RECALL

Theorem 3.3.8 **(Cofactor Expansion Theorem)**

Let A be an $n \times n$ matrix. If we multiply the elements in any row (or column) of A by their cofactors, then the sum of the resulting products is $\det(A)$. Thus,

1. If we expand along row i,

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik}.$$



2. If we expand along column j,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{k=1}^{n} a_{kj}C_{kj}.$$

§ 3.2 PROP. OF DETERMINANTS (CONT'D)

RECALL

Theorem 3.2.1 If A is an $n \times n$ upper or lower triangular matrix, then

$$\det(A) = a_{11}a_{22}a_{33}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

RECALL

P_{ij} **P1.** If B is the matrix obtained by permuting two rows of A, then

$$\det(B) = -\det(A).$$

$k=0$

$M_j(k)$ **P2.** If B is the matrix obtained by multiplying one row of A by $\boxed{\text{any}^2}$ scalar k, then

$$\det(B) = k\det(A).$$

$A_{ij}(k)$ **P3.** If B is the matrix obtained by adding a multiple of any row of A to a different row of A, then

$$\det(B) = \det(A).$$

$\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} \longrightarrow k \begin{vmatrix} a & b \\ kc & kd \end{vmatrix} \rightarrow \underbrace{k(k)}_{k^2} \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

P4. For any scalar k and $n \times n$ matrix A, we have

$$\det(kA) = k^n \det(A).$$

P5. $\det(A^T) = \det(A)$. $\rightarrow$ DETERMINANTS (ROWS ⇔ COLUMNS)

NOTATION : CP_{ij} , $CM_j(k)$, $CA_{ij}(k)$

P7. If A has a row (or column) of zeros, then $\det(A) = 0$.

P8. If two rows (or columns) of A are scalar multiples of one another, then $\det(A) = 0$.

P6. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ denote the row vectors of A. If the ith row vector of A is the sum of two row vectors, say $\mathbf{a}_i = \mathbf{b}_i + \mathbf{c}_i$, then $\det(A) = \det(B) + \det(C)$, where

$$B = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{i-1} \\ \mathbf{b}_i \\ \mathbf{a}_{i+1} \\ \vdots \\ \mathbf{a}_n \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{i-1} \\ \mathbf{c}_i \\ \mathbf{a}_{i+1} \\ \vdots \\ \mathbf{a}_n \end{bmatrix}.$$

The corresponding property is also true for columns.

$$\begin{array}{c} \\ j\text{th} \rightarrow \\ \\ \end{array} \begin{vmatrix} * & * & * & * \\ b_{j1}+c_{j1} & b_{j2}+c_{j+2} & & b_{jn}+c_{jn} \\ * & * & * & * \end{vmatrix} = \begin{vmatrix} * & * & * \\ b_{j1} & b_{j2} & b_{jn} \\ * & * & * \end{vmatrix} + \begin{vmatrix} * & * & * \\ c_{j1} & c_{j2} & c_{jn} \\ * & * & * \end{vmatrix}$$

$$\begin{vmatrix} 1 & 5 & 2 \\ c+d & a+b & e+f \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 2 \\ c & a & e \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 2 \\ d & b & f \\ 0 & 0 & 1 \end{vmatrix}$$

det BEHAVES WELL ("COMMUTES") WITH MATRIX PRODUCTS

P9. $\det(AB) = \det(A)\det(B)$.

$$A = \begin{pmatrix} 4 & 3 \\ -2 & 5 \end{pmatrix} \qquad AB = \begin{pmatrix} 4 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ -10 & 25 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \qquad \det A = (20) - (-6) = 26, \quad \det B = 10$$

$$\det(AB) = (8)(25) - (-10)(6)$$

$$= 200 + 60 = 260 = 26 \times 10 = \det A \det B$$

P10. If A is an invertible matrix, then $\det(A) \neq 0$ and $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

$$A A^{-1} = I$$

$$\Rightarrow (\det A) \cdot (\det A^{-1}) = \det I$$

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \longrightarrow \det I = 1$$

$$\Rightarrow (\det A)(\det A^{-1}) = 1 \Rightarrow \det A \neq 0$$

ALSO, $$(\det A^{-1}) = \frac{1}{(\det A)}$$

WANT : COLUMN OPERATIONS

Let $A = \begin{bmatrix} 4 & \boxed{12} & -5 & \boxed{-2} \\ -1 & -18 & 0 & 3 \\ 2 & -6 & 3 & 1 \\ 7 & 6 & -1 & -1 \end{bmatrix}$. Evaluate $\det(A)$.

$$CA_{42}(6) \longrightarrow \begin{bmatrix} 4 & 0 & -5 & -2 \\ -1 & 0 & 0 & 3 \\ 2 & 0 & 3 & 1 \\ 7 & 0 & 1 & 1 \end{bmatrix}$$

COFACTOR

$= 0$

Use ~~property P6~~ to express

$$\begin{vmatrix} a_1 + b_1 & c_1 + d_1 \\ a_2 + b_2 & c_2 + d_2 \end{vmatrix}$$

as a sum of four determinants.

$$\begin{vmatrix} a_1 & c_1 \\ a_2 + b_2 & c_2 + d_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ a_2 + b_2 & c_2 + d_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 \\ b_2 & d_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}$$

Example 3.2.12 If $A = \begin{bmatrix} \sin\phi & \cos\phi \\ -\cos\phi & \sin\phi \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, show that $\det(AB) = 1$.

$$\det A = (\sin\phi)(\sin\phi) - (-\cos\phi)(\cos\phi) = \sin^2\phi + \cos^2\phi = 1$$

$$\det B = (\cos^2\theta) + (\sin^2\theta) = 1$$

$$\det(AB) = (\det A)(\det B) = 1 \times 1 = 1$$

Example 3.2.13 Suppose that A and B are 3×3 matrices with $\boxed{\det(A) = -2}$ and $\boxed{\det(B) = 5}$, and let $D = \operatorname{diag}(-2, 1, 3)$. (Note in view of Theorem 3.2.5 that A and B are both invertible.) Compute the following:

(a) $\det(B^{-1}A^T)$.

(b) $\det(2B)$.

(c) $\det(D^2A^{-1}B)^2$.

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) $$\det(B^{-1}A^T) = (\det B^{-1})(\det A^T)$$

$$= \left[\frac{1}{\det B}\right](\det A)$$

$$= \left(\frac{1}{5}\right)(-2) = \frac{-2}{5}$$

$$
\begin{tikzcd}
B = \begin{pmatrix} 5 & 6 \\ 0 & 3 \end{pmatrix} \arrow[r, "5B"] \arrow[d, "M_1(5)"] & B'' = \begin{pmatrix} 25 & 30 \\ 0 & 15 \end{pmatrix} \\
B' = \begin{pmatrix} 25 & 30 \\ 0 & 3 \end{pmatrix} &
\end{tikzcd}
$$

$$\det B = 15$$

$$\det B' = 75 = 5 \times 15$$

$$\det B'' = 375 = 25 \times 15$$
$$= 5^2 \times 15$$

Example 3.2.13 Suppose that A and B are 3×3 matrices with $\det(A) = -2$ and $\det(B) = 5$, and let $D = \operatorname{diag}(-2, 1, 3)$. (Note in view of Theorem 3.2.5 that A and B are both invertible.) Compute the following:

(a) $\det(B^{-1}A^T)$.

(b) $\det(2B)$.

(c) $\det(D^2A^{-1}B)^2$.

$(\det kA) = k^n \det A$ (n → DIM. of A)

$\det(2B) = 2^3 \det B = 2^3 \times 5 = 40$

Example 3.2.13 Suppose that A and B are 3×3 matrices with $\det(A) = -2$ and $\det(B) = 5$, and let $D = \text{diag}(-2, 1, 3)$. (Note in view of Theorem 3.2.5 that A and B are both invertible.) Compute the following:

(a) $\det(B^{-1}A^T)$.

(b) $\det(2B)$.

(c) $\det(D^2A^{-1}B)^2$.

$A = B = X$

$\det(X^2) = (\det X)^2$

$$\det\left[(\underbrace{D^2A^{-1}B}_{X})^2\right] = \det(X^2) = (\det X)^2$$

$$\left[\det(D^2A^{-1}B)\right]^2 = \left[\det(D^2)\det(A^{-1}B)\right]^2$$

$$= \left[(\det D)^2(\det A^{-1})(\det B)\right]^2$$

$\det A = -2$

$\det B = 5$

$$D = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$$

$\det D = -6$

$$\left[(\det D)^2 \, (\det A^{-1}) \, (\det B) \right]^2$$

$$= \left[(\det D)^2 \left(\frac{1}{\det A} \right) (\det B) \right]^2$$

$$= \left[(-6)^2 \left(\frac{1}{-2} \right) (5) \right]^2$$

$$= 90^2 = 8100$$

RETURN AT
10 : 10 AM

$\boxed{\text{RETURN TO § 3.3}}$

2 WAYS TO COMPUTE DETERMINANT

(1) COFACTOR EXPANSIONS

(2) ELEMENTARY ROW / COLUMN OPERATIONS

CAN COMBINE !

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Example 3.3.11 Evaluate $\begin{vmatrix} 2 & 1 & 8 & 6 \\ \boxed{1} & 4 & 1 & 3 \\ -1 & 2 & 1 & 4 \\ 1 & 3 & -1 & 2 \end{vmatrix}$.

$A_{21}(-2)$

$A_{23}(1)$, $A_{24}(-1)$

$$\begin{vmatrix} 0 & -7 & 6 & 0 \\ 1 & 4 & 1 & 3 \\ 0 & 6 & 2 & 7 \\ 0 & -1 & -2 & -1 \end{vmatrix} = (-1) \begin{vmatrix} -7 & 6 & 0 \\ 6 & 2 & 7 \\ -1 & 2 & -1 \end{vmatrix}$$

$$(-1)\begin{vmatrix} -7 & 6 & 0 \\ 6 & 2 & 7 \\ -1 & 2 & -1 \end{vmatrix}$$

$$\Big\downarrow A_{32}(7)$$

$$(-1)\begin{vmatrix} -7 & 6 & 0 \\ -1 & 16 & 0 \\ -1 & 2 & -1 \end{vmatrix} = (-1)(-1)\begin{vmatrix} -7 & 6 \\ -1 & 16 \end{vmatrix}$$

$$= (-7)(16) - (-1)(6)$$

$$= -106$$

Example 3.3.12 Determine all values of k for which the system

$$\begin{aligned} 10x_1 + kx_2 - x_3 &= 0, \\ kx_1 + x_2 - x_3 &= 0, \\ 2x_1 + x_2 - 3x_3 &= 0, \end{aligned}$$

has nontrivial solutions.

SUFFICE TO COMPUTE

$$\begin{vmatrix} 10 & k & -1 \\ k & 1 & -1 \\ 2 & \boxed{1} & -3 \end{vmatrix} \xrightarrow{\substack{CA_{21}(-2) \\ CA_{23}(3)}} \begin{vmatrix} 10-2k & k & 3k-1 \\ k-2 & 1 & 2 \\ 0 & \boxed{1} & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 10-2k & 3k-1 \\ k-2 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 10-2k & 3k-1 \\ k-2 & 2 \end{vmatrix} = 2(10-2k) - (k-2)(3k-1)$$

$$= 20 - 4k - k^2 + 7k - 2$$

$$= 18 + 3k - 3k^2$$

$$= 3(6 + k - k^2)$$

$$\text{DET} = 3(3-k)(2+k)$$

$$\underbrace{k = 3, -2} \rightarrow$$ EQUATION HAS ∞ly MANY SOLN.

$C_{ij} \to (i,j)$ th cofactor,

$$\left(\forall i \quad \sum_{j=1}^{n} a_{ij} C_{ij} = \det A\right), \left(\forall j \quad \sum_{i=1}^{n} a_{ij} C_{ij} = \det A\right)$$

Corollary 3.3.14 If the elements in the ith row (or column) of an $n \times n$ matrix A are multiplied by the cofactors of a different row (or column), then the sum of the resulting products is zero. That is,

1. If we use the elements of row i and the cofactors of row j,

$$\sum_{k=1}^{n} a_{ik} C_{jk} = 0, \qquad i \neq j. \tag{3.3.2}$$

2. If we use the elements of column i and the cofactors of column j,

$$\sum_{k=1}^{n} a_{ki} C_{kj} = 0, \qquad i \neq j. \tag{3.3.3}$$

Corollary 3.3.14 If the elements in the ith row (or column) of an $n \times n$ matrix A are multiplied by the cofactors of a different row (or column), then the sum of the resulting products is zero. That is,

1. If we use the elements of row i and the cofactors of row j,

$$\sum_{k=1}^{n} a_{ik}C_{jk} = 0, \qquad i \neq j. \tag{3.3.2}$$

2. If we use the elements of column i and the cofactors of column j,

$$\sum_{k=1}^{n} a_{ki}C_{kj} = 0, \qquad i \neq j. \tag{3.3.3}$$

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \qquad \vec{b}_k = \begin{cases} \vec{a}_k & k \neq i \\ \vec{a}_j & k = i \end{cases}$$

$$B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$$

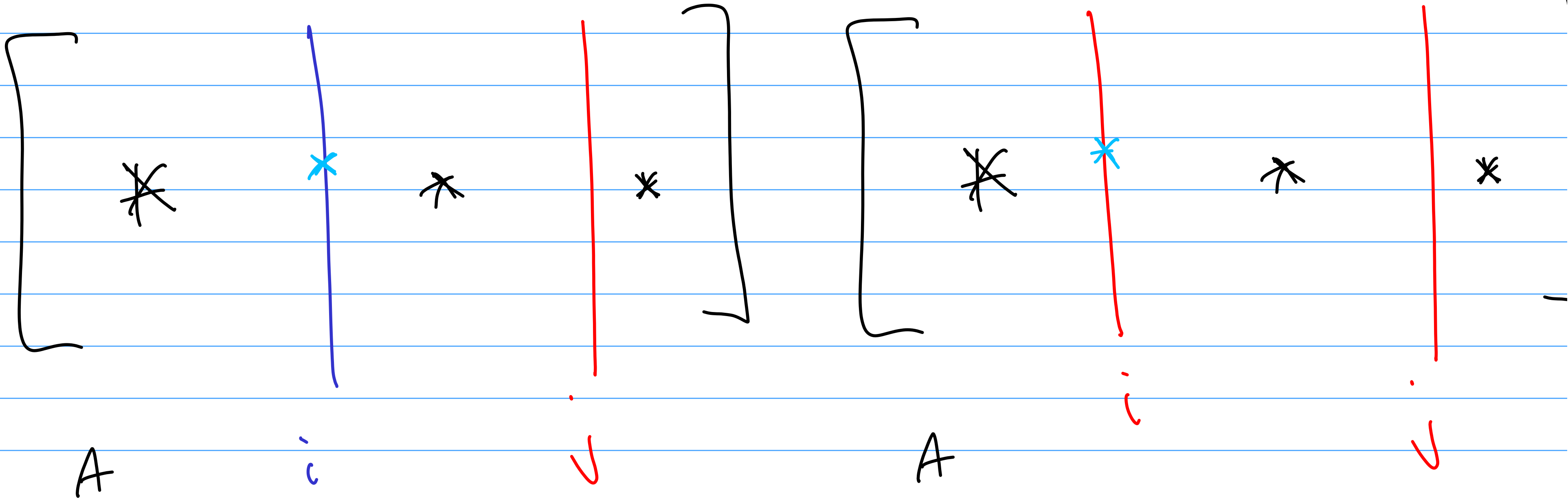
$$\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_i & \cdots & \vec{a}_j & \cdots & \vec{a}_n \end{bmatrix}$$

$$\downarrow$$

$$B = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_j & \cdots & \vec{a}_j & \cdots & \vec{a}_n \end{bmatrix}$$

$\det B = 0 \rightarrow$ COFACTOR EXPANSION ALONG i $\quad \displaystyle\sum_{k=1}^{n} a_{jk} C_{ik}$

$$0 = \sum_{k=1}^{n} b_{ik} c_{ik} = \sum_{k=1}^{n} a_{jk} \underbrace{c_{ik}}_{\text{COFACTORS OF } A}$$



$C = ((C_{ij})) \to$ MATRIX OF COFACTORS.

$\delta_{ij} \to \begin{cases} 1 & \text{IF } i=j \\ 0 & \text{OTHERWISE.} \end{cases}$

Corollary 3.3.15 Let A be an $n \times n$ matrix. If $\boxed{\delta_{ij}}$ is the Kronecker delta symbol (see Definition 2.2.21), then

$$\sum_{k=1}^{n} a_{ik} C_{jk} = \delta_{ij} \det(A), \qquad \sum_{k=1}^{n} a_{ki} C_{kj} = \delta_{ij} \det(A). \tag{3.3.4}$$

$\sum_{k=1}^{n} a_{ik} C_{jk} \to$ PRODUCT OF A & C^T

$A = (a_{ij})$

$B = (b_{ij})$

$AB = (c_{ij})$

(MATRIX MULT.) $\sum_{k=1}^{n} a_{ik} b_{kj} = \vec{a_i} \cdot \vec{b_j}$

C = MATRIX OF COFACTORS

$B = \mathrm{adj}(A) = C^T$

DEFINITION 3.3.16

If every element in an $n \times n$ matrix A is replaced by its cofactor, the resulting matrix is called the **matrix of cofactors** and is denoted M_C. The transpose of the matrix of cofactors, M_C^T, is called the **adjoint** of A and is denoted $\mathrm{adj}(A)$. Thus, the elements of $\mathrm{adj}(A)$ are

$$\mathrm{adj}(A)_{ij} = C_{ji}.$$

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= \sum_{k=1}^{n} a_{ik} C_{jk} = \delta_{ij} (\det A)$$

$$A\,(\operatorname{adj} A) = \begin{bmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \ddots \\ & & & \det A \end{bmatrix}$$

$$= (\det A) \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & 1 \\ 0 & & & 1 \\ & & & & 1 \end{bmatrix}$$

$$A\,(\operatorname{adj} A) = (\det A)\, I$$

$$A\,(\operatorname{adj} A) = (\det A)\, I$$

IF (& ONLY IF) $\det A \neq 0$

$$(A)\left[\frac{1}{\det A}\operatorname{adj}(A)\right] = I$$

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}(A)$$

Determine adj(A) if $A = \begin{bmatrix} 6 & -1 & 0 \\ 2 & -2 & 1 \\ 3 & 0 & -3 \end{bmatrix}$.

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = 6, \quad C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix} = -3$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = 6, \quad C_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 0 \\ 0 & -3 \end{vmatrix} = -3$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 6 & 0 \\ 3 & -3 \end{vmatrix} = -18, \quad C_{23} = (-1)^{2+3} \begin{vmatrix} 6 & -1 \\ 3 & 0 \end{vmatrix} = -3$$

Theorem 3.3.18 **(The Adjoint Method for Computing A^{-1})**

If $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A).$$

$t \in \mathbb{R}$

Example 3.3.20 Find A^{-1} if $A = \begin{bmatrix} -\sin t & e^{-3t}\cos t \\ \cos t & e^{-3t}\sin t \end{bmatrix}$.

$$A^{-1} = \frac{1}{\det A}\,\text{adj}\,A$$

$$\det A = (-\sin t)e^{-3t}\sin t - (\cos t)(e^{-3t}\cos t)$$

$$= -e^{-3t}\left[\sin^2 t + \cos^2 t\right]$$

$$= -e^{-3t}$$

$$\text{adj}\,A = \begin{bmatrix} e^{-3t}\sin t & -e^{-3t}\cos t \\ -\cos t & -\sin t \end{bmatrix}$$

$$A^{-1} = \frac{1}{-e^{-3t}}\begin{bmatrix} e^{-3t}\sin t & -e^{-3t}\cos t \\ -\cos t & -\sin t \end{bmatrix} = \begin{bmatrix} -\sin t & \cos t \\ e^{3t}\cos t & e^{3t}\sin t \end{bmatrix}$$

ADJOINT OF A 2x2 MATRIX

$$\text{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$C_{11} = (-1)^{1+1} |d| = d$$

$$C_{12} = (-1)^{1+2} |c| = -c$$

$$C_{21} = (-1)^{2+1} |b| = -b$$

$$C_{22} = (-1)^{2+2} |a| = a$$

$$C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$\text{adj} = C^T$$

$$= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$B_k = \begin{bmatrix} a_{11} & a_{12} & \dots & b_1 & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & b_2 & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & b_n & \dots & a_{nn} \end{bmatrix}.$$

RANK $A = n$ $\Longleftrightarrow$ $\det A \neq 0$

$$\det B_k = \sum_{j=1}^{n} b_j \, C_{jk}$$

COFACTOR OF A

Theorem 3.3.21 **(Cramer's Rule)**

If $\det(A) \neq 0$, the unique solution to the $n \times n$ system $A\mathbf{x} = \mathbf{b}$ is $(x_1, x_2, \dots, x_n)$, where

$$x_k = \frac{\det(B_k)}{\det(A)}, \qquad k = 1, 2, \dots, n. \tag{3.3.6}$$

$B_k \rightarrow$ A w/ COLUMN k REPLACED BY $\vec{b}$

$A \to$ SQUARE

$$A\vec{x} = \vec{b}$$

$$\big\downarrow$$

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

$$A\vec{x} = \vec{b} \qquad \& \qquad \det A \neq 0$$

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

$$A^{-1}A = I$$

$$x = Ix = (A^{-1}A)x = A^{-1}(A\vec{x})$$

$$\vec{x} = A^{-1}\vec{b}$$

$$= \frac{1}{\det A}(\operatorname{adj} A)\vec{b}$$

$$\vec{x} = \frac{1}{\det A}\left((\operatorname{adj} A)\,\vec{b} \right)$$

$$\downarrow$$

$$x_k = \frac{1}{\det A} \sum_{j=1}^{n} C_{jk}\, b_j = \frac{\det B_k}{\det A}$$

NEXT TIME

Use Cramer's rule to solve the system

$$\begin{aligned} -5x_1 - x_2 + 2x_3 &= 9, \\ x_1 - 2x_2 + 7x_3 &= -2, \\ 3x_1 - x_2 + x_3 &= -6. \end{aligned}$$