MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
T – 9:00 PM – 10:00 PM (ET)
F – 3:00 PM – 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1-10 ARE UPLOADED.

2. WW 04 - WAS DUE WED (13th JULY) AT 11:00 PM ET.
   WW 05 - IS DUE SAT (16th JULY) AT 11:00 PM ET.
   WW 06 - IS DUE TUE (19th JULY) AT 11:00 PM ET.

3. MIDTERM HAS BEEN GRADED. → REGRADE REQUESTS ARE OPEN TILL FRIDAY

4. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE !

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$\vdots \qquad\qquad\qquad = b_2$$

$$\vdots$$

$$a_{n1}x_1 + \cdots + a_{nn}x_n = b_n$$

$$A\vec{x} = \vec{b}$$

$$B_k = \begin{bmatrix} a_{11} & a_{12} & \ldots & b_1 & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & b_2 & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & b_n & \ldots & a_{nn} \end{bmatrix}.$$

A w/ kth COLUMN REPLACED BY $\vec{b}$

Theorem 3.3.21 **(Cramer's Rule)**

If $\det(A) \neq 0$, the unique solution to the $n \times n$ system $A\mathbf{x} = \mathbf{b}$ is $(x_1, x_2, \ldots, x_n)$, where

$$x_k = \frac{\det(B_k)}{\det(A)}, \qquad k = 1, 2, \ldots, n. \tag{3.3.6}$$

Pf sketch $\quad \vec{x} = A^{-1}\vec{b} = \frac{1}{\det A}(\operatorname{adj} A)\vec{b}$

Use Cramer's rule to solve the system

$$\begin{aligned} -5x_1 - x_2 + 2x_3 &= 9, \\ x_1 - 2x_2 + 7x_3 &= -2, \\ 3x_1 - x_2 + x_3 &= -6. \end{aligned}$$

$$A = \begin{bmatrix} -5 & -1 & 2 \\ 1 & -2 & 7 \\ 3 & -1 & 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 9 \\ -2 \\ 6 \end{bmatrix}$$

$\det A$, $\det B_1$, $\det B_2$, $\det B_3$

$$A = \begin{bmatrix} -5 & -1 & 2 \\ 1 & -2 & 7 \\ 3 & -1 & 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 9 \\ -2 \\ 6 \end{bmatrix}$$

$$\det A = \begin{vmatrix} -5 & -1 & 2 \\ 1 & -2 & 7 \\ 3 & -1 & 1 \end{vmatrix}, \quad \det B_1 = \begin{vmatrix} \boxed{\begin{matrix} 9 \\ -2 \\ 6 \end{matrix}} & \begin{matrix} -1 \\ -2 \\ -1 \end{matrix} & \begin{matrix} 2 \\ 7 \\ 1 \end{matrix} \end{vmatrix}$$

$$\det B_2 = \begin{vmatrix} \begin{matrix} -5 \\ 1 \\ 3 \end{matrix} & \boxed{\begin{matrix} 9 \\ -2 \\ 6 \end{matrix}} & \begin{matrix} 2 \\ 7 \\ 1 \end{matrix} \end{vmatrix}, \quad \det B_3 = \begin{vmatrix} \begin{matrix} -5 \\ 1 \\ 3 \end{matrix} & \begin{matrix} -1 \\ -2 \\ -1 \end{matrix} & \boxed{\begin{matrix} 9 \\ -2 \\ 6 \end{matrix}} \end{vmatrix}$$

$$\det(A) = \begin{vmatrix} -5 & -1 & 2 \\ 1 & -2 & 7 \\ 3 & -1 & 1 \end{vmatrix} = -35, \qquad \det(B_1) = \begin{vmatrix} 9 & -1 & 2 \\ -2 & -2 & 7 \\ -6 & -1 & 1 \end{vmatrix} = 65,$$

$$\det(B_2) = \begin{vmatrix} -5 & 9 & 2 \\ 1 & -2 & 7 \\ 3 & -6 & 1 \end{vmatrix} = -20, \qquad \det(B_3) = \begin{vmatrix} -5 & -1 & 9 \\ 1 & -2 & -2 \\ 3 & -1 & -6 \end{vmatrix} = -5.$$

$$x_1 = \frac{\det B_1}{\det A} = \frac{65}{35} = \frac{13}{7}$$

$$x_2 = \frac{\det B_2}{\det A} = \frac{-20}{35} = \frac{-4}{7}$$

$$x_3 = \frac{\det B_3}{\det A} = \frac{-5}{35} = -\frac{1}{7}$$

$\S 4.1$ VECTORS IN $\mathbb{R}^n$

CONSIDER $A\vec{x} = 0$

$$A = \begin{bmatrix} \boxed{1} & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix} \xrightarrow[A_{13}(-3)]{A_{12}(-2)} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_2 + 2x_3 = 0 \Rightarrow x_2 = r, x_3 = s, \; x_1 = r - 2s$

$$\vec{x} \in \{ (r - 2s, r, s) : s, r \in \mathbb{R} \}$$

$$\vec{x} = (r - 2s, r, s) = (r, r, 0) + (-2s, 0, s)$$

$$\vec{x} = (r, r, 0) + (-2s, 0, s)$$

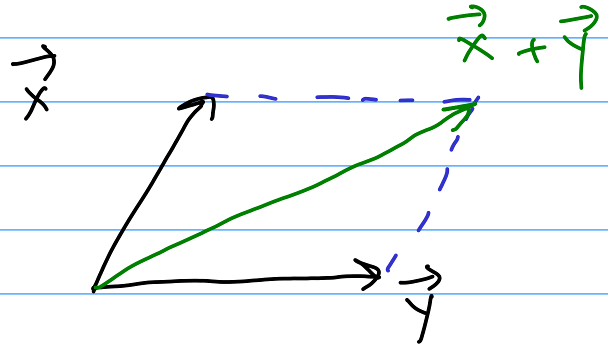
$$= r(1,1,0) + s(-2,0,1)$$

$$k\vec{x} = (kr)(1,1,0) + (ks)(-2,0,1)$$

$$\vec{x}_1 + \vec{x}_2 = (r_1 + r_2)(1,1,0) + (s_1 + s_2)(1,1,0)$$

ADDITION (ON $\mathbb{R}^n$)

$\{(x_1, \dots, x_n) : x_j \in \mathbb{R}\}$



(PARALLELOGRAM LAM)

$$\vec{x} = (x_1, \dots, x_n)$$

$$\vec{y} = (y_1, \dots, y_n)$$

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$$(1, 4) + (5, 3) = (6, 7)$$

PROPERTIES

1. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (COMMUTATIVE)

2. $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ (ASSOCIATIVE)

3. $\vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$ $\left(\vec{0} := (0, 0, \dots, 0)\right)$
[ZERO ELEMENT]

(N.B.: $\vec{0} \neq 0$, $0_{m \times n}$)

4. $\vec{x} + (-\vec{x}) = (-\vec{x}) + \vec{x} = 0$ $\left(-\vec{x} := (-x_1, \dots, -x_n)\right)$
[ADDITIVE INVERSE]

e.g. $\vec{x} = (5, 2, -1)$, $-\vec{x} = (-5, -2, 1)$, $\vec{x} + (-\vec{x}) = (0, 0, 0) = \vec{0}$

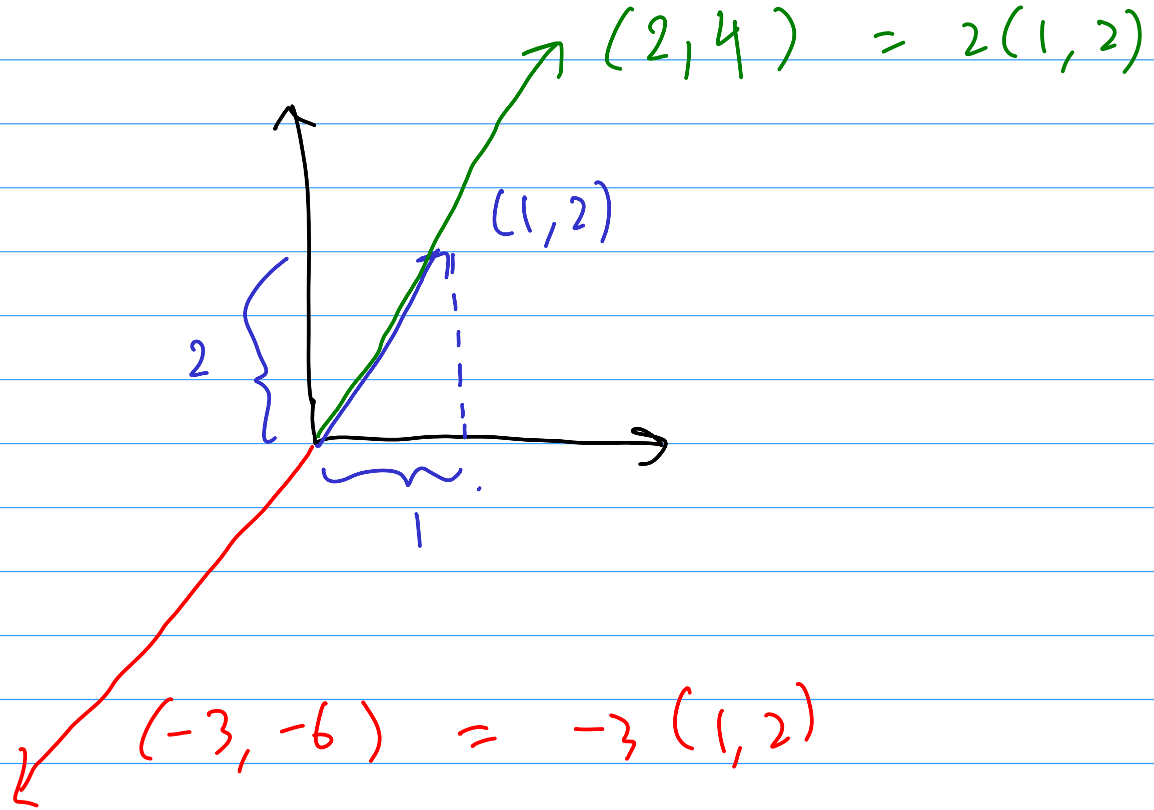
SCALAR MULTIPLICATION

$k\vec{x} = (kx_1, \ldots, kx_n)$

$k > 0 \rightarrow$ SCALING BY k
(SAME DIRECTION)

$k = 0 \rightarrow$ BECOMES 0

$k < 0 \rightarrow$ SCALING BY $|k|$,
REVERSING
DIRECTION



PROP. OF SCALAR MULT.

$$1\mathbf{x} = \mathbf{x},$$

SCALING BY 1 ≡ NO CHANGE

$$(st)\mathbf{x} = s(t\mathbf{x}),$$

SCALING MULTIPLE TIMES IS CUMULATIVE

$$r(\mathbf{x}+\mathbf{y}) = r\mathbf{x} + r\mathbf{y},$$

$$(s+t)\mathbf{x} = s\mathbf{x} + t\mathbf{x}.$$

DISTRIBUTIVITY OF SCALAR MULT. OVER ADDITION (VECTOR/SCALAR)

$\vec{v}, \vec{w} \in \mathbb{R}^6$

If $\mathbf{v} = (-7.1, 2.4, -0.1, 6, -8.3, 5.4)$ and $\mathbf{w} = (9.6, -3.3, 4, -8.1, 0, -1.7)$ are vectors in $\mathbb{R}^6$, then

$$\vec{v} + \vec{w} = (-7.1, 2.4, -0.1, 6, -8.3, 5.4) + (9.6, -3.3, 4, -8.1, 0, -1.7)$$

$$= (2.5, -0.9, 3.9, -2.1, -8.3, 3.7)$$

$$3\vec{w} = 3(9.6, -3.3, 4, -8.1, 0, -1.7) = (28.8, -9.9, 12, -24.3, 0, -5.1)$$

$$-2\vec{v} = -2(-7.1, 2.4, -0.1, 6, -8.3, 5.4) = (14.2, -4.8, 0.2, -12, 16.6, -10.8)$$

§ 4.2 DEFN. OF A VECTOR SPACE

Vector Addition: A rule for combining any two vectors in V. We will use the usual $+$ sign to denote an addition operation, and the result of adding the vectors $\mathbf{u}$ and $\mathbf{v}$ will be denoted $\mathbf{u} + \mathbf{v}$.

Real (~~or complex~~) scalar multiplication: A rule for combining each vector in V with any real (~~or complex~~) number. We will use the notation $k\mathbf{v}$ or, for emphasis, $k \cdot \mathbf{v}$, to denote the result of scalar multiplying the vector $\mathbf{v}$ by the real (~~or complex~~) number k.

R

DEFINITION 4.2.1

Let V be a nonempty set (whose elements are called vectors) on which is defined an addition operation and a scalar multiplication operation with scalars in F. We call V a **vector space over** F, provided the following ten conditions are satisfied:

A1. *Closure under addition:* For each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is also in V. We say that V is **closed under addition**.

A2. *Closure under scalar multiplication:* For each vector $\mathbf{v}$ in V and each scalar k in F, the scalar multiple $k\mathbf{v}$ is also in V. We say that V is **closed under scalar multiplication**.

A3. *Commutativity of addition:* For all $\mathbf{u}, \mathbf{v} \in V$, we have

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

A4. *Associativity of addition:* For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, we have

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

A5. *Existence of a zero vector in V:* In V there is a vector, denoted $\mathbf{0}$, satisfying

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \qquad \text{for all } \mathbf{v} \in V.$$

A6. *Existence of additive inverses in V:* For each vector $\mathbf{v} \in V$, there is a vector, denoted $-\mathbf{v}$, in V such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

A7. *Unit property:* For all $\mathbf{v} \in V$,

$$1\mathbf{v} = \mathbf{v}.$$

A8. *Associativity of scalar multiplication:* For all $\mathbf{v} \in V$ and all scalars $r, s \in F$,

$$(rs)\mathbf{v} = r(s\mathbf{v}).$$

A9. *Distributive property of scalar multiplication over vector addition:* For all $\mathbf{u}, \mathbf{v} \in V$ and all scalars $r \in F$,

$$r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}.$$

A10. *Distributive property of scalar multiplication over scalar addition:* For all $\mathbf{v} \in V$ and all scalars $r, s \in F$,

$$(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}.$$

EX. OF CLOSURE UNDER +

$$M_{2\times 2} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

$$= \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}$$

Ex. OF CLOSURE UNDER SCALAR MULT.

$$P_2(\mathbb{R}) \rightsquigarrow p(x) = ax^2 + bx + c$$

POLYNOMIALS

OF DEG ≤ 2

$$k \in \mathbb{R}, \quad kp(x) = k(ax^2 + bx + c) = (ka)x^2 + (kb)x + kc$$

E.g. OF ZERO ELEMENT.

S = SET OF CONTINUOUS FUNCTIONS FROM $[0,1] \longrightarrow \mathbb{R}$

$$f : [0,1] \longrightarrow \mathbb{R}$$

$$(f+g)(x) = f(x) + g(x)$$

$$f(x) = e^x \quad , \quad g(x) = 1$$

$$(f+g)(x) = e^x + 1$$

$$h(x) = 0$$

$$(f+h)(x) = f(x) + h(x)$$

$$= f(x)$$

$$\Rightarrow f+h = f$$

$$M_{2\times 3} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$(-A) = \begin{pmatrix} -a & -b & -c \\ -d & -e & -f \end{pmatrix}$$

$$A + (-A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

E.g. OF UNIT PROP.

$P_2(\mathbb{R})$

$$1\,(ax^2+bx+c) = ax^2+bx+c$$

$$1 \cdot p(x) = p(x)$$

BREAK
TILL

10 : 10 AM

Example 4.2.2 Let V be the set of all 2×2 matrices with real elements. Show that V, together with the usual operations of matrix addition and multiplication of a matrix by a real number, is a real vector space.

$V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$

$+ \rightarrow$ MATRIX ADD.

$\cdot \rightarrow$ MULTIPLY ALL COMPONENTS

1. CLOSURE

$$k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

2. ADDITION

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix} = \begin{pmatrix} a'+a & b'+b \\ c'+c & d'+d \end{pmatrix}$$

$$= \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

IIIly FOR ASSOCIATIVITY

EXISTENCE OF ZERO ELEMENT

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad A + O = A$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

EXISTENCE OF ADDITIVE INVERSE

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \qquad A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

a) $$1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1a & 1b \\ 1c & 1d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

b) $$(rs) A = (rs) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (rs)a & (rs)b \\ (rs)c & (rs)d \end{pmatrix}$$

$$= \begin{pmatrix} r(sa) & r(sb) \\ r(sc) & r(sd) \end{pmatrix}$$

$$= r \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix} = r \left[s \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = r(sA)$$

DISTRIBUTIVITY

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

$$r(A+B) = r\begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}$$

$$= \begin{pmatrix} r(a+a') & r(b+b') \\ r(c+c') & r(d+d') \end{pmatrix}$$

$$= \begin{pmatrix} ra + ra' & rb + rb' \\ rc + rc' & rd + rd' \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} + \begin{pmatrix} ra' & rb' \\ rc' & rd' \end{pmatrix} = rA + rB$$

$$(r+s)A = (r+s)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (r+s)a & (r+s)b \\ (r+s)c & (r+s)d \end{pmatrix}$$

$$= \begin{pmatrix} ra+sa & rb+sb \\ rc+sc & rd+sd \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} + \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix}$$

$$= rA + sA$$

Example 4.2.3 Let V be the set of all real-valued functions defined on an interval I. Define addition and scalar multiplication in V as follows. If f and g are in V and k is any real number, then $f+g$ and kf are defined by

$$(f+g)(x) = f(x) + g(x) \qquad \text{for all } x \in I,$$
$$(kf)(x) = kf(x) \qquad \text{for all } x \in I.$$

$\forall x \in I,$

(DIST. OF SCALAR ADDITION)

$$[\underbrace{(r+s)f}](x) = (r+s)f(x) = rf(x) + sf(x)$$
$$= (rf)(x) + (sf)(x)$$
$$= (\underbrace{rf + sf})(x)$$

$$\Rightarrow (r+s)f = rf + sf$$

f, g, h

(ASSOC.)

$x \in I,$

$$\Big[\underbrace{(f+g)+h}\Big](x) = (f+g)(x) + h(x)$$
$$= \big[f(x) + g(x)\big] + h(x)$$
$$= f(x) + \big[g(x) + h(x)\big]$$
$$= f(x) + (g+h)(x)$$
$$= \Big[\underbrace{f + (g+h)}\Big](x)$$

$$(f+g)+h = f + (g+h)$$

Example 4.2.4 Let V be the set of all polynomials with real coefficients and of degree 2 or less, together with the usual operations of polynomial addition and multiplication of a polynomial by a real number. Show that V is a real vector space.

$$P_2(\mathbb{R}) = \{ ax^2 + bx + c \ : a, b, c \in \mathbb{R} \}$$

$$= \{ p(x) \ : \ \deg p \leq 2 \}$$

$+$: POLYNOMIAL ADDITION

$\cdot$: $k \cdot (ax^2 + bx + c) = (ka)x^2 + (kb)x + (kc)$

ZERO ELEMENT : $q(x) = 0$

$$p(x) + q(x) = p(x)$$

ADDITIVE INVERSES

$p(x) = x^2 + x + 1$

$q(x) = -x^2 - x - 1$

$p(x) + q(x) = 0$

IN GEN: $p(x) = ax^2 + bx + c$

$q(x) = -ax^2 - bx - c$

$\Big\} \rightarrow p(x) + q(x) = 0$

Theorem 4.2.7 Let V be a vector space over $\cancel{F}$. $\mathbb{R}$

1. The zero vector is unique.
2. $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$.
3. $k\mathbf{0} = \mathbf{0}$ for all scalars $k \in \cancel{F}$. $\mathbb{R}$
4. The additive inverse of each element in V is unique.
5. For all $\mathbf{v} \in V$, $-\mathbf{v} = (-1)\mathbf{v}$.
6. If k is a scalar and $\mathbf{v} \in V$ such that $k\mathbf{v} = \mathbf{0}$, then either $k = 0$ or $\mathbf{v} = \mathbf{0}$.

Pf of 1. 0_1 & 0_2 s.t. $\boxed{v + 0_1 = v}$ I , $v \in V$

$\boxed{v + 0_2 = v}$ II

$0_1 + 0_2 = 0_1$ ($v = 0_1$ IN II) ; $0_2 + 0_1 = 0_2$ ($v = 0_2$ IN I)

$\forall v, w \in V,$ $\quad v + w = w + v \quad$ (COMM. OF $+$)

$v = 0_1 \ , \ w = 0_2$

$0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2$

$\boxed{0_1 = 0_2}$ $\rightarrow$ UNIQUE !

$(r+s)\vec{v} = r\vec{v} + s\vec{v}$

Theorem 4.2.7 Let V be a vector space over F.

1. The zero vector is unique.
2. $\boxed{0\mathbf{v} = \mathbf{0} \text{ for all } \mathbf{v} \in V.}$
3. $k\mathbf{0} = \mathbf{0}$ for all scalars $k \in F$.
4. The additive inverse of each element in V is unique.
5. For all $\mathbf{v} \in V$, $-\mathbf{v} = (-1)\mathbf{v}$.
6. If k is a scalar and $\mathbf{v} \in V$ such that $k\mathbf{v} = \mathbf{0}$, then either $k = 0$ or $\mathbf{v} = \mathbf{0}$.

$0 = 0 + 0$ in $\mathbb{R}$

$0\vec{v} = (0+0)\vec{v} = 0\vec{v} + 0\vec{v}$

(DIST. OF SCALAR ADDITION)

$$0\vec{v} = 0\vec{v} + 0\vec{v}$$

ADD $-[0\vec{v}]$ (ADDITIVE INVERSE OF $0\vec{v}$)

LHS $\Rightarrow$ $0\vec{v} + (-0\vec{v}) = \vec{0}$

RHS $\Rightarrow$ $(0\vec{v} + 0\vec{v}) + (-0\vec{v}) = 0\vec{v} + \underbrace{[0\vec{v} + (-0\vec{v})]}_{\vec{0}}$

ASSOC.

$$= 0\vec{v} + \vec{0} = 0\vec{v}$$

$$0\vec{v} = \vec{0}$$

Theorem 4.2.7 Let V be a vector space over F.

1. The zero vector is unique.
2. $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$.
3. $\boxed{k\mathbf{0} = \mathbf{0} \text{ for all scalars } k \in F.}$
4. The additive inverse of each element in V is unique.
5. For all $\mathbf{v} \in V$, $-\mathbf{v} = (-1)\mathbf{v}$.
6. If k is a scalar and $\mathbf{v} \in V$ such that $k\mathbf{v} = \mathbf{0}$, then either $k = 0$ or $\mathbf{v} = \mathbf{0}$.

$\vec{0} + \vec{0} = \vec{0}$, DIST. OF VECTOR ADDITION

v, w_1, w_2 (w_j IS AN INVERSE FOR v)

$$\underbrace{(w_1 + v)}_{=0} + w_2 = w_1 + v + w_2 = w_1 + \underbrace{(v + w_2)}_{=0}$$

$0 + w_2 = w_2$

$= w_1 + 0 = w_1$

Theorem 4.2.7 Let V be a vector space over F.

$(1 \cdot v = v)$

$(r+s) \cdot v = rv + sv$

1. The zero vector is unique.
2. $\boxed{0\mathbf{v} = \mathbf{0} \text{ for all } \mathbf{v} \in V.}$
3. $k\mathbf{0} = \mathbf{0}$ for all scalars $k \in F$.
4. The additive inverse of each element in V is unique.
5. For all $\mathbf{v} \in V$, $-\mathbf{v} = (-1)\mathbf{v}$.
6. If k is a scalar and $\mathbf{v} \in V$ such that $k\mathbf{v} = \mathbf{0}$, then either $k = 0$ or $\mathbf{v} = \mathbf{0}$.

$$(-\vec{v}) = \underbrace{(-1)}_{\text{SCALAR}} \vec{v}$$

ADDITIVE INVERSE OF V

$$\vec{v} + (-1)\vec{v} = 1 \cdot \vec{v} + (-1) \cdot \vec{v}$$

$$= [\underbrace{1 + (-1)}] \cdot \vec{v} \quad \text{(DIST.)}$$

$$= [0] \cdot \vec{v} = \vec{0}$$

$$\vec{v} + (-1)\vec{v} = \vec{0}$$

$(-1)\vec{v}$ IS AN ADDITIVE INVERSE OF V !

$$-\vec{v} = (-1)\vec{v}$$

Theorem 4.2.7 Let V be a vector space over F.

1. The zero vector is unique.
2. $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$.
3. $k\mathbf{0} = \mathbf{0}$ for all scalars $k \in F$.
4. The additive inverse of each element in V is unique.
5. For all $\mathbf{v} \in V$, $-\mathbf{v} = (-1)\mathbf{v}$.
6. If k is a scalar and $\mathbf{v} \in V$ such that $k\mathbf{v} = \mathbf{0}$, then either $k = 0$ or $\mathbf{v} = \mathbf{0}$.

<u>CLAIM</u>: $k\vec{v} = \vec{0}$ & $k \neq 0 \implies \vec{v} = \vec{0}$

$s = 1/k$, $\quad s(k\vec{v}) = (sk)\vec{v} = (1)\cdot\vec{v} = \vec{v}$

(ASSOC. OF MULT.)

$\vec{v} = s(k\vec{v}) = s(\vec{0}) = \vec{0}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad -A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

$$= \begin{pmatrix} (-1)a & (-1)b \\ (-1)c & (-1)d \end{pmatrix}$$

$$= (-1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (-1)A$$