MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
T – 9:00 PM – 10:00 PM (ET)
F – 3:00 PM – 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEB PAGE $\boxed{\text{https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html}}$

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1 – 11 ARE UPLOADED.

2. WW 05 – WAS DUE SAT (16th JULY) AT 11:00 PM ET.
   WW 06 – IS DUE TUE (19th JULY) AT 11:00 PM ET.
   WW 07 – IS DUE SAT (23rd JULY) AT 11:00 PM ET.

3. MIDTERM 2 IS ON MONDAY (25th JULY) ⇝ SCHEDULER

4. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE !

RECALL

VECTOR ADDITION

$(V, +, \cdot)$ (OVER $\mathbb{R}$)

SCALAR MULTIPLICATION

DEFINITION 4.2.1

Let V be a nonempty set (whose elements are called vectors) on which is defined an addition operation and a scalar multiplication operation with scalars in F. We call V a **vector space over** F, provided the following ten conditions are satisfied:

A1. *Closure under addition:* For each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is also in V. We say that V is **closed under addition**.

A2. *Closure under scalar multiplication:* For each vector $\mathbf{v}$ in V and each scalar k in F, the scalar multiple $k\mathbf{v}$ is also in V. We say that V is **closed under scalar multiplication**.

A3. *Commutativity of addition:* For all $\mathbf{u}, \mathbf{v} \in V$, we have

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

A4. *Associativity of addition:* For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, we have

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

A5. *Existence of a zero vector in V:* In V there is a vector, denoted $\mathbf{0}$, satisfying

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \qquad \text{for all } \mathbf{v} \in V.$$

A6. *Existence of additive inverses in V:* For each vector $\mathbf{v} \in V$, there is a vector, denoted $-\mathbf{v}$, in V such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

A7. *Unit property:* For all $\mathbf{v} \in V$,

$$1\mathbf{v} = \mathbf{v}.$$

A8. *Associativity of scalar multiplication:* For all $\mathbf{v} \in V$ and all scalars $r, s \in F$,

$$(rs)\mathbf{v} = r(s\mathbf{v}).$$

A9. *Distributive property of scalar multiplication over vector addition:* For all $\mathbf{u}, \mathbf{v} \in V$ and all scalars $r \in F$,

$$r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}.$$

A10. *Distributive property of scalar multiplication over scalar addition:* For all $\mathbf{v} \in V$ and all scalars $r, s \in F$,

$$(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}.$$

e.g. $\mathbb{R}^n \longrightarrow \{(x_1, \dots, x_n) : x_j \in \mathbb{R}\}$

$P_n(\mathbb{R}) \longrightarrow \{p(x) \in \mathbb{R}[x] : \deg p \le n\}$

$M_{m \times n}(\mathbb{R}) \longrightarrow \{$ $m \times n$ MATRICES OVER $\mathbb{R}\}$

$V(I) = \{f : I \to \mathbb{R}\}$ $\qquad I \to$ INTERVAL

RECALL

THEOREMS ABOUT VECTOR SPACES.

Theorem 4.2.7 Let V be a vector space over $\cancel{\mathbb{F}}$. $\mathbb{R}$

1. The zero vector is unique.
2. $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$.
3. $k\mathbf{0} = \mathbf{0}$ for all scalars $k \in \cancel{\mathbb{F}}$. $\mathbb{R}$
4. The additive inverse of each element in V is unique.
5. For all $\mathbf{v} \in V$, $-\mathbf{v} = (-1)\mathbf{v}$.
6. If k is a scalar and $\mathbf{v} \in V$ such that $k\mathbf{v} = \mathbf{0}$, then either $k = 0$ or $\mathbf{v} = \mathbf{0}$.

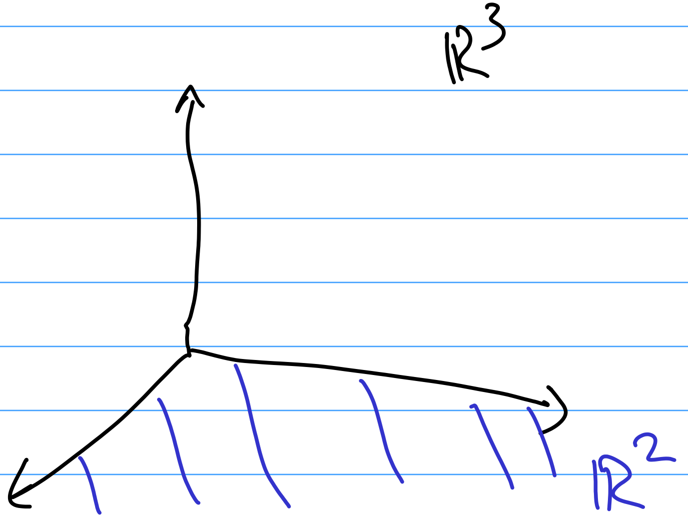
$\boxed{\text{§ 4.3} \quad \text{SUBSPACES}}$

DEFINITION 4.3.1

Let S be a nonempty subset of a vector space V. If S is itself a vector space under the same operations of addition and scalar multiplication as used in V, then we say that S is a **subspace** of V.

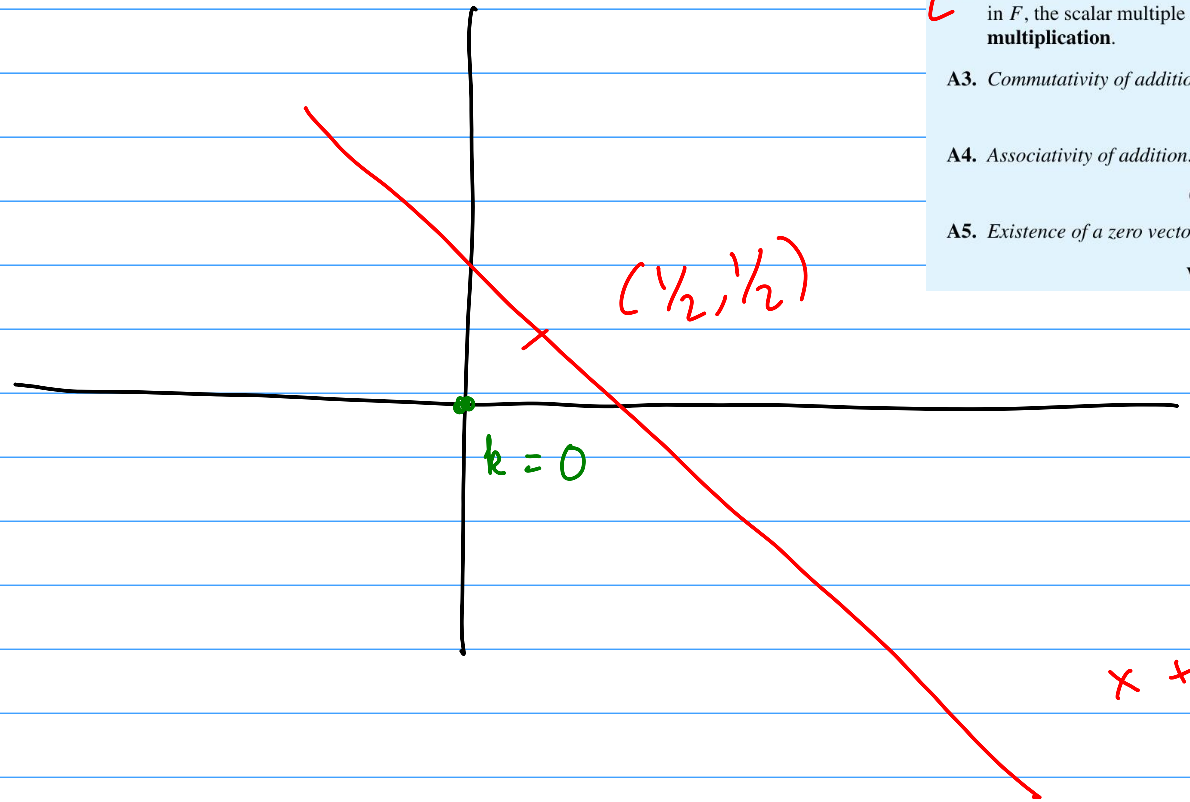
$$\mathbb{R}^2 \subseteq \mathbb{R}^3$$

$$(x_1, x_2) \longrightarrow (x_1, x_2, 0)$$



NOT A SUBSPACE

$\mathbb{R}^2$



$k \neq 1$

$x + y = 1$

DEFINITION 4.2.1

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A2. *Closure under scalar multiplication:* For each vector $\mathbf{v}$ in V and each scalar k in F, the scalar multiple $k\mathbf{v}$ is also in V. We say that V is **closed under scalar multiplication**.

A3. *Commutativity of addition:* For all $\mathbf{u}, \mathbf{v} \in V$, we have

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

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$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

A5. *Existence of a zero vector in V:* In V there is a vector, denoted $\mathbf{0}$, satisfying

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \qquad \text{for all } \mathbf{v} \in V.$$

Theorem 4.3.2 Let S be a nonempty subset of a vector space V. Then S is a subspace of V if and only if S is closed under the operations of addition and scalar multiplication in V.

PF $\rightarrow$ CHECK A3 — A10 ARE TRUE IF ANY SUBSET IF A1 & A2.

Example 4.3.3 Let S denote the set of all real solutions to the following linear system of equations:

$$\begin{aligned} x_1 - \quad 4x_2 + \quad 6x_3 &= 0, \\ -3x_1 + 10x_2 - 10x_3 &= 0. \end{aligned}$$

Express S in set notation and verify that S is a subspace of $\mathbb{R}^3$.

$$\left[\begin{array}{ccc|c} \boxed{1} & -4 & 6 & 0 \\ -3 & 10 & -10 & 0 \end{array}\right]$$

$$\Big\downarrow A_{12}(3)$$

$$\left[\begin{array}{ccc|c} 1 & -4 & 6 & 0 \\ 0 & -2 & 8 & 0 \end{array}\right]$$

$$\downarrow M_2(-1/2)$$

$$\left[\begin{array}{ccc|c} 1 & -4 & 6 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right]$$

$r = 2$
$n = 3$
deg of freedom $= 3 - 2$
$= 1$

$$x_1 - 4x_2 + 6x_3 = 0$$

$$x_2 - 4x_3 = 0$$

$x_3 = t$

$x_2 = 4t$

$x_1 = 4x_2 - 6x_3 = 16t - 6t = 10t$

$(x_1, x_2, x_3) = (10t, 4t, t)$

SET OF SOLUTIONS

$$S = \{ (10t, 4t, t) \in \mathbb{R}^3 \mid t \in \mathbb{R} \}$$

CLAIM: $S \subseteq \mathbb{R}^3$ (SUBSPACE)

A1: CLOSURE UNDER $+$.

$$(10t, 4t, t) + (10s, 4s, s) = (10t + 10s, 4t + 4s, t + s)$$

$$= (10(t+s), 4(t+s), t+s) \in S \quad (u = t + s)$$

A2: CLOSURE UNDER SCALAR MULT.

$k \in \mathbb{R}$, $(10t, 4t, t) \in S$

$$k \cdot (10t, 4t, t) = (10kt, 4kt, kt) \in S \quad (v = kt)$$

Example 4.3.4 Verify that $S = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = (r, -3r + 1),\ r \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^2$.

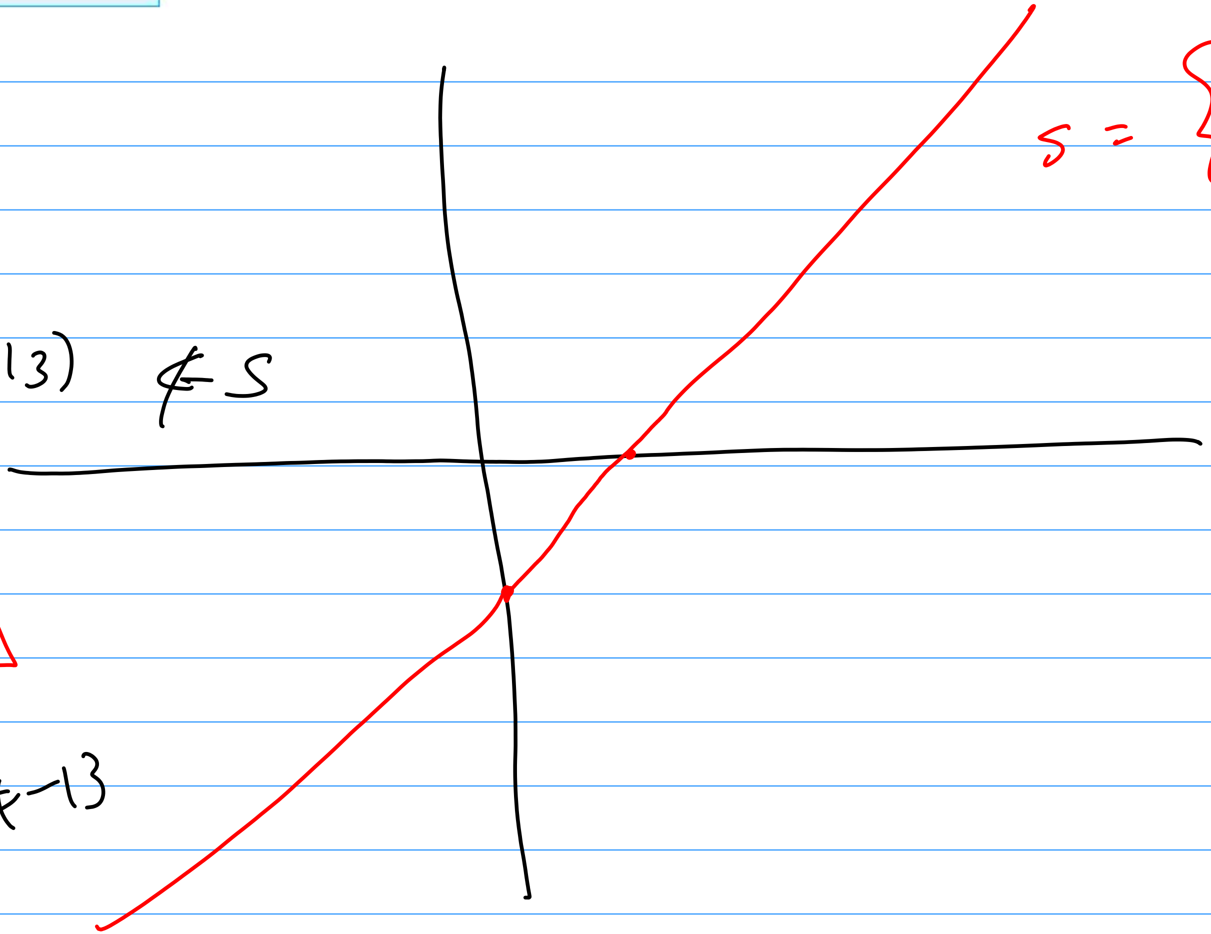
$\vec{u} = (3, -8)$

$\vec{v} = (2, -5)$

$\vec{u} + \vec{v} = (5, -13) \notin S$

$$\boxed{\begin{aligned} r &= 5 \\ -3r + 1 &= -13 \end{aligned}}$$

$-15 + 1 = -14 \neq -13$



$S = \{(r, -3r+1) : r \in \mathbb{R}\}$

Zero Vector Check

> If a subset S of a vector space V fails to contain the zero vector $\mathbf{0}$, then it cannot form a subspace.

IMP : THIS CAN ONLY TELL YOU THAT S IS NOT A SUBSPACE.

$\vec{0} = (0,0)$

Example 4.3.5 Let $V = \mathbb{R}^2$, and let

$$S_1 = \{(x, x-1) : x \in \mathbb{R}\} \quad \text{and} \quad S_2 = \{(x, x^2) : x \in \mathbb{R}\}.$$

NOT A VECTOR SPACE.

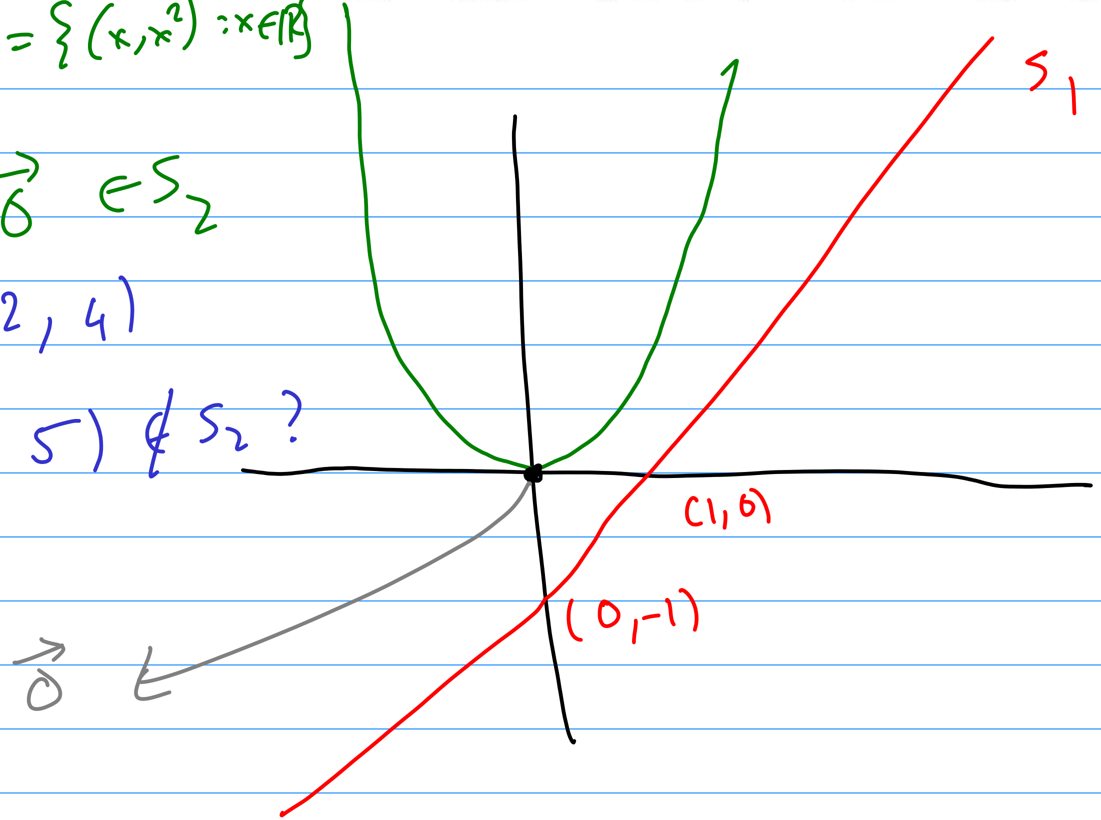
$S_2 = \{(x,x^2) : x \in \mathbb{R}\}$

$S_1 = \{(x, x-1) : x \in \mathbb{R}\}$

$\vec{0} \in S_2$

$(1,1) + (2,4)$

$= (3,5) \notin S_2$?



$\vec{0} \notin S_1$

(NOT A SUBSPACE $\to$ FAILS ZERO VECTOR CHECK)

Example 4.3.6 Let S denote the set of all real skew-symmetric $n \times n$ matrices. Verify that S is a subspace of $M_n(\mathbb{R})$.

SKEW-SYMMETRIC : $A^T = -A$

A1 CLOSURE UNDER ADDITION, $A, B \in S$

$A^T = -A, \quad B^T = -B$

$(A+B)^T = A^T + B^T = (-A) + (-B)$

$= -(A+B)$

$\Rightarrow A+B \in S$

A2 : CLOSURE UNDER SCALAR MULT.

$k \in \mathbb{R} \qquad A \in S. \qquad [A^T = -A]$

$$(kA)^T = k(A^T) = k(-A) = -kA$$

$\Rightarrow kA \in S$

$\Rightarrow S =$ SKEW-SYMMETRIC IS A SUBSPACE OF $n \times n$ MATRICES.

Example 4.3.7 Let $V = M_{2\times 3}(\mathbb{R})$, and let S denote the set of all elements of V for which the entries in each column sum to zero. Show that S is a subspace of V.

$$V = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} : a, \dots, f \in \mathbb{R} \right\}$$

S = ALL COLUMN SUMS ARE ZERO

$$S = \left\{ \begin{bmatrix} a & b & c \\ -a & -b & -c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

CLOSURE +

$$: \begin{bmatrix} a & b & c \\ -a & -b & -c \end{bmatrix} + \begin{bmatrix} a' & b' & c' \\ -a' & -b' & -c' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' & c+c' \\ -(a+a') & -(b+b') & -(c+c') \end{bmatrix}$$

CLOSURE $\cdot$

$k \in \mathbb{R}$

$$\begin{bmatrix} a & b & c \\ -a & -b & -c \end{bmatrix}$$

$$k \begin{bmatrix} a & b & c \\ -a & -b & -c \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ -ka & -kb & -kc \end{bmatrix} \in S$$

$S \rightarrow$ CLOSED UNDER $+$ & $\cdot \mathbb{R}$

$\Rightarrow$ S IS A SUBSPACE.

Example 4.3.8 Let V be the vector space of all real-valued functions defined on an interval $[a, b]$, and let S denote the set of all functions f in V that satisfy $f(a) = f(b)$. Verify that S is a subspace of V.

$$V = \{ f : [a,b] \longrightarrow \mathbb{R} \}$$

$$(f+g)(x) = f(x) + g(x)$$
$$(kf)(x) = k\, f(x)$$

A1: $f, g \in S$ $\left[\begin{array}{l} f(a) = f(b), \\ g(a) = g(b) \end{array} \right]$

$$(f+g)(a) = f(a) + g(a)$$
$$= f(b) + g(b) = (f+g)(b) \Rightarrow f+g \in S$$

A2: $k \in \mathbb{R}$, $f \in S$ $[f(a) = f(b)]$, $(kf)(a) = kf(a) = kf(b) = (kf)(b) \Rightarrow kf \in S$

$$P_2(\mathbb{R}) \to \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$$

Example 4.3.9 Let V be the vector space $P_2(\mathbb{R})$, fix $r \in \mathbb{R}$, and let S denote the set of polynomials $p(x) \in V$ such that $p(r) = 0$. Express S in set notation and verify that S is a subspace of V.

$$S = \{p \in P_2(\mathbb{R}) : p(1) = 0\}$$

TRY THIS : A1, A2 ARE SATISFIED.

Theorem 4.3.10 Let V be a vector space with zero vector $\mathbf{0}$. Then $S = \{\mathbf{0}\}$ is a subspace of V.

$$\left.\begin{aligned} \vec{0} + \vec{0} &= \vec{0} \\ k \cdot \vec{0} &= \vec{0} \end{aligned}\right\} \rightarrow \{\vec{0}\} \text{ SS } \quad \text{CLOSED!}$$

(TRIVIAL SUBSPACE)

Theorem 4.3.11 Let A be an $m \times n$ matrix. The solution set of the homogeneous system of linear equations $A\mathbf{x} = \boxed{\mathbf{0}}$ is a subspace of ~~$\mathbb{C}^n$ (or $\mathbb{R}^n$ if the solutions are real).~~

$\mathbb{R}^n$

m EQUATIONS IN n VARIABLES

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

$$S = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = 0 \}$$

① $\vec{x}, \vec{y} \in S$, $\vec{x} + \vec{y} \in S$

② $k \in \mathbb{R}$, $\vec{x} \in S$, $k\vec{x} \in S$

DEFINITION 4.3.12

SPECIAL SUBSPACE ASSOCIATED W/ A.

Let A be an $m \times n$ matrix. The solution set to the corresponding homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is called the $\boxed{\textbf{null space of } A}$ and is denoted nullspace(A). Thus,

$$\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

$\mathbb{R}^n$

$\vec{0} = A(\vec{0}) = \vec{b} \neq \vec{0}$

NOT A SUBSPACE

$$\boxed{A\vec{x} = \vec{b} \qquad \vec{b} \neq 0}$$

N.B.: $b = 0$!

Example 4.3.13 Let V denote the vector space of all real-valued functions that are defined on an interval I, and let $C^k(I)$ denote the set of all functions that are continuous and have (at least) k continuous derivatives on the interval I, for a fixed positive integer k. Show that $C^k(I)$ is a subspace of V.

$$V = \{ f : I \longrightarrow \mathbb{R} \}$$

$$C^k(I) = \{ f \in V : f \text{ IS CONT., } k \text{ TIMES DIFF'BLE, \& HAS CONTINUOUS DERIVATIVES} \}$$

FOLLOWS FROM

(1) SUM OF CONTINUOUS / DIFF'BLE FUNCTIONS IS CONT./DIFF.

(2) SCALAR MULTIPLE OF CONT/DIFF IS CONT./DIFF.

Theorem 4.3.14 The set of all solutions to the homogeneous linear differential equation

$$y'' + a_1(x)y' + a_2(x)y = 0$$

on an interval I is a vector space. V = SET OF SOLUTION

$$V \subseteq C^2(I)$$

A1 & A2

<u>A1</u> : $y_1, y_2 \in V$ $y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0$

$y_2'' + a_1(x)y_2' + a_2(x)y_2 = 0$

$$(y_1+y_2)'' + a_1(x)(y_1+y_2)' + a_2(x)(y_1+y_2)$$

$$(y_1+y_2)'' + a_1(x)(y_1+y_2)' + a_2(x)(y_1+y_2)$$

$$= y_1'' + y_2'' + a_1(x)y_1' + a_1(x)y_2' + a_2(x)y_1 + a_2(x)y_2$$

$$= \left[y_1'' + a_1(x)y_1' + a_2(x)y_1\right] + \left[y_2'' + a_1(x)y_2' + a_2(x)y_2\right]$$

$$= 0 + 0 = 0$$

$$\Rightarrow y_1 + y_2 \in V$$

$y_1 \in V$, $k \in \mathbb{R} \Rightarrow ky_1 \in V$

$$\boxed{\begin{gathered} y_1, y_2 \in V \\ y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0 \\ y_2'' + a_1(x)y_2' + a_2(x)y_2 = 0 \end{gathered}}$$

V IS CLOSED & $V \subseteq C^2(I)$

$\Rightarrow$ V IS A SUBSPACE OF $C^2(I)$

$\Rightarrow$ V IS A VECTOR SPACE.

BREAK TILL

10 : 15 AM

§ 4.4 SPANNING SETS

OLD THM:

Theorem 2.2.9 If $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ is an $m \times n$ matrix and $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is a column n-vector, then

$$A\mathbf{c} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n. \tag{2.2.2}$$

$$A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$$

$$\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$A\vec{c} = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$= c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n$$

If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are column m-vectors and $c_1, c_2, \ldots, c_n$ are scalars, then an expression of the form

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n$$

is called a **linear combination** of the column vectors. Therefore, from Equation (2.2.2), we see that the vector $A\mathbf{c}$ is obtained by taking a linear combination of the column vectors of A.

DEFINITION 4.4.1

If *every* vector in a vector space V can be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_k$, we say that V is **spanned** or **generated** by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and call the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ a **spanning set** for V. In this case, we also say that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ **spans** V.

$$\mathbb{R}^3 \rightarrow (x, y, z) = x\underbrace{(1,0,0)}_{\vec{e}_1} + y\underbrace{(0,1,0)}_{\vec{e}_2} + z\underbrace{(0,0,1)}_{\vec{e}_3}$$

$$\overbrace{\{e_1, e_2, e_3\}}^{\text{SPANNING SET}} \quad \underline{\text{SPANS}} \quad \mathbb{R}^3$$

STANDARD SPANNING SET $\{\vec{e}_1, \ldots, \vec{e}_n\}$ in $\mathbb{R}^n$

$$(x_1, \ldots, x_n) = x_1(1,0,\ldots) + x_2(0,1,0,\ldots) + \cdots + x_n(0,\ldots,0,1)$$

$$= x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

$$\mathbb{R}^n = \mathrm{Span}(\vec{e}_1, \ldots, \vec{e}_n)$$

$\vec{e}_1 = (1,0)$

$\vec{e}_2 = (0,1)$

Example 4.4.2 Show that $\mathbb{R}^2$ is spanned by the vectors

$$\mathbf{v}_1 = (1, 1) \quad \text{and} \quad \mathbf{v}_2 = (2, -1).$$

$\vec{v} \in \mathbb{R}^2 \qquad \vec{v} = (x, y)$

$(\exists c_1, c_2 \in \mathbb{R}) \qquad \vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2$

$(x, y) = c_1 (1, 1) + c_2 (2, -1) = (c_1 + 2c_2, c_1 - c_2)$

$$\boxed{\begin{aligned} c_1 + 2c_2 &= x \\ c_1 - c_2 &= y \end{aligned}} \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & x \\ 1 & -1 & y \end{array}\right]$$

$$\det\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = (-1) - (2) = -3 \neq 0$$

ALWAYS HAS A SOLN.

$\vec{v} \in \mathbb{R}^2 \Rightarrow \vec{v} = c_1 \vec{v_1} + c_2 \vec{v_2}$

$(\exists c_1, c_2)$

$\Rightarrow \mathbb{R}^2$ IS SPANNED BY

$v_1 = (1,1)$, $v_2 = (2,-1)$

Example 4.4.3 Determine whether the vectors $\mathbf{v}_1 = (1, -3, 6)$, $\mathbf{v}_2 = (1, -4, 2)$, and $\mathbf{v}_3 = (-2, 10, 4)$ span $\mathbb{R}^3$.

$\vec{v} \in \mathbb{R}^3 \qquad \vec{v} = (x_1, x_2, x_3)$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{v}$$

$$c_1(1, -3, 6) + c_2(1, -4, 2) + c_3(-2, 10, 4) = (x_1, x_2, x_3)$$

$$\begin{aligned} c_1 + c_2 - 2c_3 &= x_1 \\ -3c_1 - 4c_2 + 10c_3 &= x_2 \\ 6c_1 + 2c_2 + 4c_3 &= x_3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 1 & -2 & x_1 \\ -3 & -4 & 10 & x_2 \\ 6 & 2 & 4 & x_3 \end{array}\right]$$

$\text{rank } A = 2 < n = 3$

$$\left[\begin{array}{ccc|c} \multicolumn{3}{c}{} & \\ 1 & 1 & -2 & x_1 \\ 0 & 1 & -4 & -3x_1 - x_2 \\ \underbrace{0 \quad 0 \quad 0}_{A} & & & \boxed{-18x_1 - 4x_2 + x_3} \end{array}\right]. \quad A^{\#}$$

$-18x_1 - 4x_2 + x_3 \to$ $= 0$ (∞ SOLN)
$\neq 0$ (NO SOLN.)

① $\text{rank } A = \text{rank } A^{\#} < n \rightsquigarrow$ $n - \text{rank}$ FREE VAR & ∞ SOLN

② $\text{rank } A < \text{rank } A^{\#} \rightsquigarrow$ NO SOLN.

$\therefore v_1, v_2, v_3$ IS NOT SPANNING $\mathbb{R}^3$
($\because (1,1,1) \neq c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$)

∞ SOLN :

$$S = \{(x_1, x_2, x_3) \mid -18x_1 - 4x_2 + x_3 = 0\} \subseteq \mathbb{R}^3$$

$\to$ SUBSPACE OF $\mathbb{R}^3$

v_1, v_2, v_3 SPANS S.

$$S = \text{Span}\{v_1, v_2, v_3\}$$

Theorem 4.4.4 Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ spans $\mathbb{R}^n$ if and only if, for the matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$, the linear system $A\mathbf{c} = \mathbf{v}$ is consistent for every $\mathbf{v}$ in $\mathbb{R}^n$.

$$A c = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n$$

$\vec{v} = A \vec{c}$ FOR SOME CHOICE OF $\vec{c}$

$$A^{\#} = \left[\begin{array}{ccc|c} \vec{v}_1 & \cdots & \vec{v}_k & \vec{v} \end{array} \right]$$

Example 4.4.5 Verify that $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and $A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ span $M_2(\mathbb{R})$.

$$\left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$$A \in M_2(\mathbb{R}) \quad , \quad A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$A = c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_4$$

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 + c_4 & c_2 + c_3 + c_4 \\ c_3 + c_4 & c_4 \end{bmatrix}$$

$$c_1 + c_2 + c_3 + c_4 = x_1$$

$$c_2 + c_3 + c_4 = x_2$$

$$c_3 + c_4 = x_3$$

$$c_4 = x_4$$

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1 \neq 0$$

BACK - SUBSTITUTION

Example 4.4.6 Determine a spanning set for $P_2(\mathbb{R})$, the vector space of all polynomials of degree 2 or less.

$$P_2(\mathbb{R}) = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$$

$$\left.\begin{matrix} x^2 \\ x \\ 1 \end{matrix}\right\} \qquad \{1, x, x^2\}$$

$\downarrow$

SPANNING

$$ax^2 + bx + c = \underbrace{a(x^2) + b(x) + c(1)}_{\text{LINEAR COMB. OF } \{x^2, x, 1\}}$$

The Linear Span of a Set of Vectors

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in a vector space V. Forming all possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ generates a subset of V called the **linear span** of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, denoted $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. We have

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \{\mathbf{v} \in V : \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \,,\ c_1, c_2, \ldots, c_k \in F\}.$$

<u>NOTE</u>: $\{v_1, \ldots, v_k\}$ IS SPANNING

IF & ONLY IF

$$\text{span}\{v_1, \ldots, v_k\} = V$$

$\subseteq V$.

Theorem 4.4.7 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in a vector space V. Then $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a subspace of V.

A1: $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$, $\vec{u}, \vec{w} \in \text{Span}(\ldots)$

$\vec{w} = c_1'\vec{v}_1 + c_2' v_2 + \cdots + c_n' v_n$

$\vec{u} + \vec{w} = (c_1 + c_1')\vec{v}_1 + (c_2 + c_2')v_2 + \ldots + (c_n + c_n')v_n$

$\in \text{Span}(\vec{v}_1, \ldots, \vec{v}_n)$

A2

$\vec{u} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$

$k\vec{u} = (kc_1)\vec{v}_1 + \ldots + (kc_n)\vec{v}_n \in \text{Span}(v_1, \ldots, v_n)$

Example 4.4.8 If $V = \mathbb{R}^2$ and $\mathbf{v}_1 = (-1, 1)$, determine span$\{\mathbf{v}_1\}$.

$$\text{span}(\vec{v}_1) = \{ \vec{v} \in \mathbb{R}^2 : \vec{v} = c_1 \vec{v}_1 \}$$

$$c_1 (-1, 1) = (-c_1, c_1)$$

$$\text{span}(\vec{v}_1) = \{ (-c_1, c_1) \in \mathbb{R}^2 : c_1 \in \mathbb{R} \}$$

$$= \{ (-t, t) \in \mathbb{R}^2 : t \in \mathbb{R} \}$$

Example 4.4.9 If $V = \mathbb{R}^3$ and $\mathbf{v}_1 = (1, 0, 1)$ and $\mathbf{v}_2 = (0, 1, 1)$, determine the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. Does $\mathbf{w} = (1, 1, -1)$ lie in this subspace?

$$\begin{aligned}\operatorname{Span}(\vec{v}_1, \vec{v}_2) &= \{ c_1 \vec{v}_1 + c_2 \vec{v}_2 : c_1, c_2 \in \mathbb{R} \} \\ &= \{ c_1 (1,0,1) + c_2 (0,1,1) : c_1, c_2 \in \mathbb{R} \} \\ &= \{ (c_1, c_2, c_1 + c_2) : c_1, c_2 \in \mathbb{R} \}\end{aligned}$$

$\vec{w} \in \operatorname{Span}(\vec{v}_1, \vec{v}_2)$?

INCONSISTENT

$(1, 1, -1) = (c_1, c_2, c_1 + c_2)$?

$$\boxed{\begin{aligned} c_1 \quad\quad &= 1 \\ c_2 &= 1 \\ c_1 + c_2 &= -1 \end{aligned}}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{array}\right]$$

$\Rightarrow \vec{w} \notin \operatorname{Span}(v_1, v_2)$

Example 4.4.10 Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in $M_2(\mathbb{R})$. Determine $\text{span}\{A_1, A_2, A_3\}$.

$$\text{Span}(A_1, A_2, A_3) = \{c_1 A_1 + c_2 A_2 + c_3 A_3 : c_1, c_2, c_3 \in \mathbb{R}\}$$

$$= \left\{c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ c_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_3 \end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} : c_1, c_2, c_3 \in \mathbb{R}\right\}$$

2×2 SYMMETRIC MATRICES

TRANSPOSE ($A^T = A$)
SYMMETRIC

Example 4.4.11 Determine the subspace of $P_2(\mathbb{R})$ spanned by

$$p_1(x) = 1 + 3x, \qquad p_2(x) = x + x^2,$$

and decide whether $\{p_1(x), p_2(x)\}$ is a $\boxed{\text{spanning set for } P_2(\mathbb{R})}$. $\rightarrow$ NO !

$$\text{Span}\{p_1, p_2\} = \{c_1 p_1(x) + c_2 p_2(x) : c_1, c_2 \in \mathbb{R}\}$$

$$= \{c_1(1+3x) + c_2(x+x^2) : c_1, c_2 \in \mathbb{R}\}$$

$$= \{c_2 x^2 + (3c_1 + c_2)x + c_1 : c_1, c_2 \in \mathbb{R}\}$$

$$ax^2 + bx + c \longrightarrow \begin{array}{l} a = c_2 \\ b = 3c_1 + c_2 \\ c = c_1 \end{array}$$

$$\begin{aligned} c_2 &= a \\ 3c_1 + c_2 &= b \\ c_1 &= c \end{aligned} \quad \rightsquigarrow \quad \left[\begin{array}{cc|c} 0 & 1 & a \\ 3 & 1 & b \\ 1 & 0 & c \end{array}\right]$$

$$\Big\downarrow \text{ PERMUTATIONS EROs}$$

$$\left[\begin{array}{cc|c} 1 & 0 & c \\ 0 & 1 & a \\ 3 & 1 & b \end{array}\right]$$

$$\xleftarrow{\text{ADD. EROs}}$$

$$\overbrace{\left[\begin{array}{cc|c} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}}_{\text{rank } 2} & \begin{matrix} c \\ a \\ b-3c-a \end{matrix} \end{array}\right]}^{\text{rank} \to = 2 \ (b-3c-a=0), \quad \to = 3 \ (b-3c-a \neq 0)}$$

SKIP

Example 4.4.12 Find a spanning set for the vector space V of all 3×3 skew-symmetric matrices.

Example 4.4.13 Find a spanning set for the null space of the matrix $A = \begin{bmatrix} -1 & 5 & 3 \\ 2 & -10 & -6 \end{bmatrix}$.

$$N = \{ \vec{x} \in \mathbb{R}^3 : A\vec{x} = \vec{0} \}$$

$$\left[\begin{array}{ccc|c} -1 & 5 & 3 & 0 \\ 2 & -10 & -6 & 0 \end{array}\right]$$

$$\Big\downarrow M_1(-1)$$

$$\left[\begin{array}{ccc|c} 1 & -5 & -3 & 0 \\ 2 & -10 & -6 & 0 \end{array}\right] \xrightarrow{A_{12}(-2)} \left[\begin{array}{ccc|c} 1 & -5 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 - 5x_2 - 3x_3 = 0$$

$n = 3$

rank $= r = 1$

deg of freedom $= 3 - 1$

$= 2$

$$x_2 = s$$

$$x_3 = t$$

$$x_1 = 5x_2 + 3x_3 = 5s + 3t$$

$$(x_1, x_2, x_3) = (5s + 3t, s, t)$$

NULLSPACE $(A) = \{ (5s+3t, s, t) : s, t \in \mathbb{R} \}$

$$(5s+3t, s, t) = (5s, s, 0) + (3t, 0, t)$$
$$= s(5, 1, 0) + t(3, 0, 1)$$

$$N(A) = \{ s(5,1,0) + t(3,0,1) : s, t \in \mathbb{R} \}$$
$$= \operatorname{Span}((5,1,0), (3,0,1))$$

$(5,1,0), (3,0,1) \rightsquigarrow$ SPANS THE NULLSPACE.