MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
T – 9:00 PM – 10:00 PM (ET)
F – 3:00 PM – 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1 - 12 ARE UPLOADED.

2. WW 06 - IS DUE ~~TUE (19th JULY)~~ AT 11:00 PM ET.
WED (20th JULY)

WW 07 - IS DUE ~~SAT (23rd JULY)~~ AT 11:00 PM ET.
SUN (24th JULY)

2.5 NEXT WEEK → WW 08 & 09 DUE WED (27th JULY)

3. MIDTERM 2 IS ON MONDAY (25th JULY) ~> SCHEDULER

4. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE !

§4.5 LINEAR DEPENDENCE & LINEAR INDEPENDENCE

Theorem 4.5.2 Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of at least two vectors in a vector space V. If one of the vectors in the set is a linear combination of the other vectors in the set, then that vector can be deleted from the given set of vectors and the linear span of the resulting set of vectors will be the same as the linear span of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

SPANS $\mathbb{R}^2$

$(1,1) = (1,0) + (0,1)$

$$S = \{ (1,0), (0,1), (1,1) \} \qquad S \subseteq \mathbb{R}^2$$

$$\mathbb{R}^2 \ni (x,y) = \frac{y}{2}(1,1) + \frac{y}{2}(0,1) + \left(x - \frac{y}{2}\right)(1,0)$$

$$S' = \{(1,0), (0,1)\} \longrightarrow (x,y) = x(1,0) + y(0,1)$$

$$S'' = \{(1,1),(0,1)\}$$

$$(x,y) = x(1,1) + (y-x)(0,1)$$

$$S''' = \{(1,0),(1,1)\}$$

$$S_0 = \{(1,0)\} \longrightarrow (0,1) \in \text{span}(S_0)$$

$v_0 \in \operatorname{Span}(v_1, \dots, v_k)$

$\Rightarrow \operatorname{Span}(v_0, \dots, v_k) = \operatorname{Span}(v_1, \dots, v_k)$

$c_0 v_0 + c_1 v_1 + \dots + c_k v_k$

$c_0 [b_1 v_1 + \dots + b_k v_k] + c_1 v_1 + \dots + c_k v_k$

$= (c_0 b_1 + c_1) v_1 + \dots + (c_0 b_k + c_k) v_k$

$\in \operatorname{Span}(v_1, \dots, v_k)$

DEFINITION 4.5.3

A finite nonempty set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is said to be **linearly dependent** if there exist scalars $c_1, c_2, \ldots, c_k$, *not all zero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Such a nontrivial linear combination of vectors is sometimes referred to as a **linear dependency** among the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

$$\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k \quad \text{(TRIVIAL)}$$

$\{(0,1), (1,0), (1,1)\}$

$$c_1(1,0) + c_2(0,1) + c_3(1,1) = \vec{0}$$

$c_1 = c_2 = -c_3 = 1$

$$(1,0) + (0,1) - (1,1) = \vec{0}$$

L.I.

DEFINITION 4.5.4

A finite, nonempty set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in a vector space V is said to be **linearly independent** if the *only* values of the scalars $c_1, c_2, \dots, c_k$ for which

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

are $c_1 = c_2 = \cdots = c_k = 0$.

L.I. = NOT (L.D.)

Q. WHEN IS $\{v\}$ L.I.?

L.D?

FIND A v, SUCH THAT, $\exists\, c \neq 0$

WITH $cv = \vec{0}$ $\longrightarrow$ ANS = $v = \vec{0}$.

$1 \cdot \vec{0} = \vec{0}$

$\{\vec{0}\}$ IS L.D.

$\vec{v} \neq \vec{0} \Rightarrow \{\vec{v}\}$ IS L.I. $\left(\because c\vec{v} = \vec{0} \Rightarrow \begin{array}{l} c = 0 \\ \text{OR } \cancel{\vec{v} = \vec{0}} \end{array} \right)$

Theorem 4.5.5 A set consisting of a single vector $\mathbf{v}$ in a vector space V is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$. Therefore, any set consisting of a single *nonzero* vector is linearly independent.

$(1,1) = (1,0) + (0,1) \Rightarrow (1,0) + (0,1) - (1,1) = \vec{0}$

Theorem 4.5.6 Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of at least two vectors in a vector space V. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent if and only if at least one of the vectors in the set can be expressed as a linear combination of the others.

$\exists\, c_0, c_1, \ldots, c_k$ NOT ALL ZERO, WITH

$$c_0 v_0 + c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

WLOG, $c_0 \neq 0$

$$\Rightarrow c_0 v_0 = -c_1 v_1 - c_2 v_2 - \cdots - c_k v_k$$

$$v_0 = \frac{1}{c_0}(c_0 v_0) = -\frac{1}{c_0}\left[c_1 v_1 + c_2 v_2 + \cdots + c_k v_k\right]$$

$$= v_1\left[-\frac{c_1}{c_0}\right] + v_2\left[-\frac{c_2}{c_0}\right] + \cdots + v_k\left[-\frac{c_k}{c_0}\right]$$

Example 4.5.7 Let V be the vector space of all functions defined on an interval I. If

$$f_1(x) = 1, \qquad f_2(x) = 2\sin^2 x, \qquad f_3(x) = -5\cos^2 x,$$

IS THERE A LINEAR DEPENDENCE?

$$\sin^2 x + \cos^2 x = 1 \longrightarrow \text{PYTHAGORAS' THEOREM.}$$

$$\sin^2 x + \cos^2 x - 1 = 0$$

$$\frac{1}{2} f_2(x) + \left(-\frac{1}{5}\right) f_3(x) - f_1(x) = 0$$

Q. WHEN IS $\{v_1, v_2\}$ L.I. ?

AT LEAST ONE OF $c_1, c_2 \neq 0$

$c_1 v_1 + c_2 v_2 = 0$ L.D ?

WLOG, $c_1 \neq 0$, CASE I : $c_2 = 0$, $c_1 v_1 = 0 \Rightarrow v_1 = 0$

CASE II : $c_2 \neq 0$, $v_2 = \left(\frac{-c_1}{c_2}\right) v_1 \rightarrow$ VECTORS ARE PROPOR.

ANS : L.D. $\Longleftrightarrow$ $v_1 \propto v_2$ (PROPORTIONAL)

Q. $\vec{0} \in S$. CAN S BE L.I.?

NO!

$S = \{ \vec{v}_0, \dots, \vec{v}_k \}$

$\vec{v}_0 = \vec{0}$.

$$\underset{1}{\underset{\parallel}{c_0}} \vec{v}_0 + \underset{0}{\underset{\parallel}{c_1}} \vec{v}_1 + \dots + \underset{0}{\underset{\parallel}{c_k}} \vec{v}_k = \vec{0}$$

$\{ \vec{0} \}$ $\quad 2 \cdot \vec{0} = \vec{0}$

$\{ \vec{0}, \vec{v}_1 \}$ $\quad 2 \cdot \vec{0} + 0 \cdot \vec{v}_1 = 2 \cdot \vec{0} + \vec{0} = \vec{0}$

Proposition 4.5.8 Let V be a vector space.

1. Any set of *two* vectors in V is linearly dependent if and only if the vectors are proportional.
2. Any set of vectors in V containing the zero vector is linearly dependent.

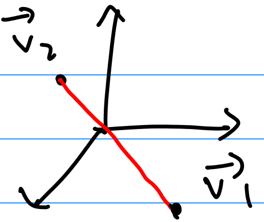
$-2(1,2,-9)$

$\shortparallel$

$\vec{v}_2 = -2\vec{v}_1$

$2\vec{v}_1 + \vec{v}_2 = 0$

Example 4.5.9 If $\mathbf{v}_1 = (1, 2, -9)$ and $\mathbf{v}_2 = (-2, -4, 18)$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent in $\mathbb{R}^3$, since $\mathbf{v}_2 = -2\mathbf{v}_1$. Geometrically, $\mathbf{v}_1$ and $\mathbf{v}_2$ lie on the same line. □



Example 4.5.10 If $\mathbf{v}_1 = (2, 4)$, $\mathbf{v}_2 = (-3, 1)$, and $\mathbf{v}_3 = (-1, 5)$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent in $\mathbb{R}^2$ since $\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$, but no two of these three vectors are proportional. □

$\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = (2 + (-3) - (-1), 4 + 1 - 5)$

$(0, 0) = \vec{0}$

Example 4.5.11 If $A_1 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $A_3 = \begin{bmatrix} 2 & 5 \\ -3 & 2 \end{bmatrix}$, then $\{A_1, A_2, A_3\}$ is linearly dependent in $M_2(\mathbb{R})$, since it contains the zero vector from $M_2(\mathbb{R})$. □

$$0 \cdot A_1 + 1 \cdot A_2 + 0 \cdot A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{2\times 2}$$

GENERAL CASE?

Example 4.5.12 If $\mathbf{v}_1 = (3, -1, 2)$, $\mathbf{v}_2 = (-9, 5, -2)$, and $\mathbf{v}_3 = (-9, 9, 6)$, show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent in $\mathbb{R}^3$, and determine the linear dependency relationship.

WANT TO SHOW, $\exists\, c_1, c_2, c \in \mathbb{R}$ (NOT ALL 0) s.t.

$$c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}$$

$$c_1(3,-1,2) + c_2(-9,5,-2) + c_3(-9,9,6) = (0,0,0)$$

$$\Rightarrow (3c_1 - 9c_2 - 9c_3,\ -c_1 + 5c_2 + 9c_3,\ 2c_1 - 2c_2 + 6c_3) = (0,0,0)$$

$$(3c_1 - 9c_2 - 9c_3,\ -c_1 + 5c_2 + 9c_3,\ 2c_1 - 2c_2 + 6c_3) = (0,0,0)$$

$$3c_1 - 9c_2 - 9c_3 = 0$$

$$-c_1 + 5c_2 + 9c_3 = 0$$

$$2c_1 - 2c_2 + 6c_3 = 0$$

$$A\vec{c} = \vec{0} \iff A = \begin{bmatrix} 3 & -9 & -9 \\ -1 & 5 & 9 \\ 2 & -2 & 6 \end{bmatrix}$$

$\vec{x} = A^{-1}\vec{0} = \vec{0}$ $\iff A\vec{x} = \vec{0}$ Rank $A = 3 \iff \det A \neq 0$

rank A = rank $[A \mid \vec{0}]$

TO SHOW $\vec{c} \neq \vec{0}$, s.t. $A\vec{c} = \vec{0}$

IT SUFFICES TO SHOW

1. $\det A = 0$; OR
2. $\text{Rank } A < 3$

POLY. OF DEG ≤ 3

Example 4.5.13 Determine whether the set of polynomials $\{p_1(x), p_2(x), p_3(x), p_4(x)\}$ is linearly dependent or linearly independent in $\boxed{P_3(\mathbb{R})}$, where

$$p_1(x) = 1-4x^3, \qquad p_2(x) = 2+2x, \qquad p_3(x) = 1-x^2+2x^3, \qquad p_4(x) = 2x-x^3.$$

$$\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_4 \end{bmatrix} \neq \vec{0}$$

$$c_1 p_1 + c_2 p_2 + c_3 p_3 + c_4 p_4 = 0 \longleftrightarrow$$

$$c_1(1-4x^3) + c_2(2+2x) + c_3(1-x^2+2x^3) + c_4(2x-x^3) = 0$$

$$\Rightarrow \underset{= 0}{(c_1 + 2c_2 + c_3)} + x\underset{= 0}{(2c_2 + 2c_4)} + x^2 \overset{= 0}{(-c_3)} + x^3\underset{= 0}{(-4c_1 + 2c_3 - c_4)} = 0$$

$$
\begin{aligned}
c_1 + 2c_2 + c_3 &= 0 \\
2c_2 + 2c_4 &= 0 \\
-c_3 &= 0 \\
-4c_1 + 2c_3 - c_4 &= 0
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ -4 & 2 & 0 & -1 \end{bmatrix}
\xrightarrow{A_{14}(4)}
\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 10 & 4 & -1 \end{bmatrix}
$$

$$\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 10 & 4 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 10 & 4 & -1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} -1 & 0 \\ 4 & -1 \end{vmatrix} = 2$$

$\det A \neq 0 \iff \text{rank } A = 4 \iff A\vec{z} = \vec{0}$ HAS A UNIQUE SOLN.

$\iff \vec{z} = 0$ IS THE ONLY CASE

$\therefore$ $p_1, \ldots, p_4$ is L.I.

BREAK TILL

10 : 15 AM

Corollary 4.5.14 Any nonempty, finite set of linearly dependent vectors in a vector space V contains a linearly independent subset that has the same linear span as the given set of vectors.

$\{v_1, \dots, v_k\}$ $\qquad v_1 \in \text{span}(v_2, \dots, v_k)$

$\Rightarrow \text{span}(v_1, \dots, v_k) = \text{span}(v_2, \dots, v_k)$

DROP SOME VECTORS

$\Rightarrow \{v_1', \dots, v_j'\} \longrightarrow$ L.I.

$\text{span}(v_1', \dots, v_j') = \text{span}(v_1, \dots, v_k)$

Pf. $\{v_1, \ldots, v_k\}$ IS L.I. ✓

SUPPOSE $\{v_1, \ldots, v_k\}$ IS L.D.

WLOG

$\Rightarrow v_k \in \text{span}\{v_1, \ldots, v_{k-1}\}$

$\Rightarrow \text{span}\{v_1, \ldots, v_k\} = \text{span}\underbrace{\{v_1, \ldots, v_{k-1}\}}_{}$

L.I. ✓

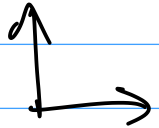
SUPPOSE NOT. $\{v_1, \ldots, v_{k-1}\}$ IS L.D.

WLOG. , $v_{k-1} \in \text{span}\{v_1, \ldots, v_{k-2}\}$

$\{\cancel{(1,1)}, (0,1), (1,0)\}$

$c_1(0,1) + c_2(1,2) = 0$

$\Rightarrow c_1 = c_2 = 0$



Example 4.5.15 Let $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-1, 1, 4)$, $\mathbf{v}_3 = (3, 3, 2)$, and $\mathbf{v}_4 = (-2, -4, -6)$. Determine a linearly independent set of vectors that spans the same subspace of $\mathbb{R}^3$ that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ does.

$$\boxed{v_4 = -2v_1} \longrightarrow \text{L.D.}$$

$$\text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_1, v_2, v_3, v_4\}$$

$$\boxed{v_3 = 2v_1 - v_2} = 2(1,2,3) - (-1,1,4)$$

$$= (2,4,6) + (1,-1,-4)$$

$$= (3,3,2)$$

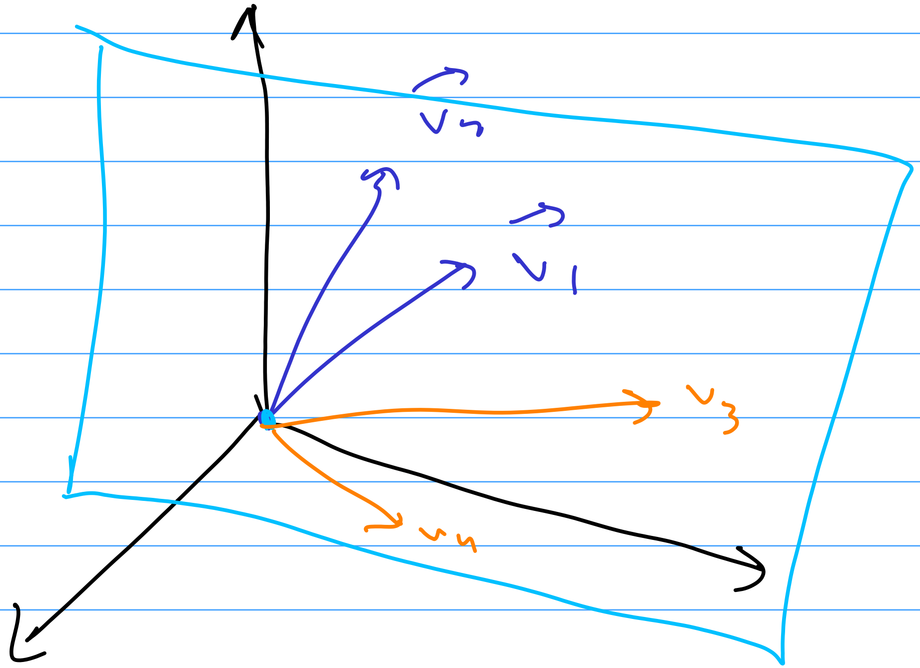
$\mathrm{span}\{v_1, v_2\} = \mathrm{span}\{v_1, v_2, v_3\} = \mathrm{span}\{v_1, v_2, v_3, v_4\}$

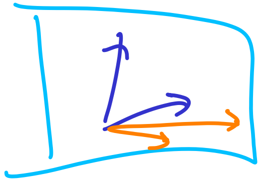
$(\because v_3 \in \mathrm{span}\{v_1, v_2\})$ $\quad (v_4 \in \mathrm{span}\{.\})$

$\{v_1, v_2\}$ IS L.I.

Pf. $v_2 = k\, v_1$ (PROP.)

$(-1, 1, 4) = k(1, 2, 3) \longrightarrow$ CANNOT HAPPEN.


$\vec{v}_2$
$\vec{v}_1$
$\vec{v}_3$
$\vec{v}_4$



$\vec{c} \in \mathbb{R}^k$

Theorem 4.5.16 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$ and $A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k]$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent if and only if the linear system $A\mathbf{c} = \mathbf{0}$ has a nontrivial solution for $\mathbf{c}$.

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_k \vec{v}_k = \vec{0}$$

$$\Rightarrow \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_k \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \vec{0}$$

$k \times 1$ MATRIX

$A \vec{c} = \vec{0}$

$n \times k$ MATRIX

1 SOLN. $(\vec{c} = \vec{0})$

∞ SOLN.

Corollary 4.5.17 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$ and $A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k]$.

1. If $k > n$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent.
2. If $k = n$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent if and only if $\det(A) = 0$.

$A\vec{c} = \vec{0}$

$\text{Rank } A \leq n < \underbrace{k}_{\text{\# of variables}}$

$\Rightarrow \infty$ MANY SOLNS.

$A \to$ SQUARE

L.D $\iff \det A = 0$

L.I. $\iff \det A \neq 0$

Example 4.5.18 Determine whether the given vectors are linearly dependent or linearly independent in $\mathbb{R}^4$.

$\mathbb{R}^4$ $(n=4)$, $k=5$

(a) $\mathbf{v}_1 = (1, 3, -1, 0)$, $\mathbf{v}_2 = (2, 9, -1, 3)$, $\mathbf{v}_3 = (4, 5, 6, 11)$, $\mathbf{v}_4 = (1, -1, 2, 5)$, $\mathbf{v}_5 = (3, -2, 6, 7)$.

(b) $\mathbf{v}_1 = (1, 4, 1, 7)$, $\mathbf{v}_2 = (3, -5, 2, 3)$, $\mathbf{v}_3 = (2, -1, 6, 9)$, $\mathbf{v}_4 = (-2, 3, 1, 6)$.

$k > n \Rightarrow$ ∞ LY MANY SOLNS

To $[\vec{v}_1 \; \ldots \; \vec{v}_5]\vec{c} = \vec{0}$

$\Rightarrow \vec{v}_1, \ldots, \vec{v}_5$ I L.D.

$k = n$ $\quad \det[\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3 \; \vec{v}_4] = \begin{vmatrix} 1 & 3 & 2 & -2 \\ 4 & -5 & -1 & 3 \\ 1 & 2 & 6 & 1 \\ 7 & 3 & 9 & 6 \end{vmatrix}$

§ 4.6 BASES & DIMENSION

GOAL: FORMALIZE NOTION OF DEGREES OF FREEDOM / DIMENSION.

e.g. $y'' + y = 0$ $\longrightarrow$ ORDER

GENERAL SOL'N

$$y = \underbrace{c_1 \cos x + c_2 \sin x}_{\text{2 DEGREES OF FREEDOM.}} \in \mathrm{Span}(\cos x, \sin x)$$

DEFINITION 4.6.1

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is called a **basis**[5] for V if

(a) The vectors are linearly independent.

(b) The vectors span V.

(ROUGHLY $\simeq$ BOTH PROP FROM TODAY & YESTERDAY.

2x2 MATRICES

Example 4.6.2 Determine the standard basis for $M_2(\mathbb{R})$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{E_{11}} + b\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{E_{12}} + c\underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{E_{21}} + d\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{E_{22}}$$

($E_{ij} \to$ 1 AT ROW i, COLUMN j
& 0 OTHERWISE)

$$\mathrm{span}\{E_{11}, E_{12}, E_{21}, E_{22}\} = M_2(\mathbb{R})$$

$\{E_{11}, E_{12}, E_{21}, E_{22}\}$ IS L.I.

Pf.

$$c_1 E_{11} + c_2 E_{12} + c_3 E_{21} + c_4 E_{22} = 0_{2\times 2}$$

$$\Rightarrow c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow c_j = 0$$

(STANDARD) BASIS.

Example 4.6.3 Determine $\boxed{\text{a}}$ basis for $P_2(\mathbb{R})$. $\rightarrow \{1, x, x^2\}$

$\rightsquigarrow$ deg ≤ 2

$\{ ax^2 + bx + c : a, b, c \in \mathbb{R} \}$

$x^2, x, 1$

$ax^2 + bx + c = a(x^2) + b(x) + c(1)$

$\text{Span}\{1, x, x^2\} = P_2(\mathbb{R})$

$c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 = 0$

$c_1 = 0, c_2 = 0, c_3 = 0 \Rightarrow \{1, x, x^2\}$ IS L.I

Remark More generally, consider the vector space of all $m \times n$ matrices with real entries, $M_{m \times n}(\mathbb{R})$. If we let E_{ij} denote the $m \times n$ matrix with value 1 in the (i, j)-position and zeros elsewhere, then one can show routinely that

$$\begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix}$$

$$\{E_{ij} : 1 \leq i \leq m, \ 1 \leq j \leq n\}$$

is a basis for $M_{m \times n}(\mathbb{R})$, and it is the **standard basis** for $M_{m \times n}(\mathbb{R})$.

Remark More generally, the reader can check that a basis for the vector space of all polynomials of degree n or less, $P_n(\mathbb{R})$, is

$$\{1, x, x^2, \ldots, x^n\}.$$

This is the **standard basis** for $P_n(\mathbb{R})$.

$\exists$ A FINITE BASIS.

Theorem 4.6.4 If a finite-dimensional vector space has a basis consisting of n vectors, then any set of more than n vectors is linearly dependent.

$\{\vec{v}_1, \ldots, \vec{v}_n\} \rightarrow$ BASIS

$\{\vec{u}_1, \ldots, \vec{u}_k\} \quad k > n$

ONLY MANY SOLN.

$$\boxed{c_1 \vec{u}_1 + c_2 \vec{u}_2 + \ldots + c_k \vec{u}_k = 0}$$

$\vec{u}_j \in \text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\} = V$

$\vec{u}_j = a_{1j} \vec{v}_1 + a_{2j} \vec{v}_2 + \ldots + a_{nj} \vec{v}_j$

$$c_1(a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + \dots + a_{n1}\vec{v}_n)$$

$$+ c_2(a_{12}\vec{v}_1 + \dots + a_{n2}\vec{v}_n)$$

$$+$$

$$\vdots$$

$$+ c_k(a_{1k}\vec{v}_1 + \dots + a_{nk}\vec{v}_n) = \vec{0}$$

$$\Rightarrow \vec{v}_1(\underbrace{c_1 a_{11} + c_2 a_{12} + \dots + c_k a_{1k}}_{=0})$$

$$+ \vec{v}_2(\underbrace{c_1 a_{21} + c_2 a_{22} + \dots + c_k a_{2k}}_{=0}) = \vec{0}$$

$$+ \dots + \vec{v}_n(\underbrace{c_1 a_{n1} + \dots + c_k a_{nk}}_{=0})$$

$$a_{11}c_1 + a_{12}c_2 + a_{13}c_3 + \cdots + a_{1k}c_k = 0$$

$$\vdots \qquad\qquad \vdots \quad = 0$$

$$a_{n1}c_1 + a_{n2}c_2 + \cdots + a_{nk}c_k = 0$$

$$A\vec{c} = \vec{0}$$

$$A = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le k}}$$

$\operatorname{rank} A \le n < k$

$\Rightarrow \infty$LY MANY $\vec{c}$

s.t. $A\vec{c} = \vec{0}$

Corollary 4.6.5 All bases in a finite-dimensional vector space V contain the same number of vectors.

PF. SUPPOSE NOT.

$m > n$, $B_1 \to m$ ELEMENTS

$B_2 \to n$ ELEMENTS

BUT, PVS. THM $\Rightarrow$ B_1 IS DEPENDEN

CONTRADICTION!

Corollary 4.6.6 If a finite-dimensional vector space V has a basis consisting of n vectors, then every spanning set for V must contain at least n vectors.

$S \to m$ ELEMENTS. $m < n$.

$S' \subseteq S$, S' IS L.I.

$\mathrm{span}(S') = \mathrm{span}\, S = V$ $\Big\} \to S'$ IS A BASIS

DEFINITION 4.6.7

The **dimension** of a finite-dimensional vector space V, written $\dim[V]$, is the number of vectors in any basis for V. If V is the trivial vector space, $V = \{\mathbf{0}\}$, then we define its dimension to be zero.

4.6.6 $\rightarrow$ $|S| < \dim V \Rightarrow$ S IS NOT SPANNING

4.6.4 $\rightarrow$ $|S| > \dim V \Rightarrow$ S IS LINEARLY DEPENDENT.

e_1, e_2, e_3

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

Example 4.6.8 It follows from our examples earlier in this section that $\dim[\mathbb{R}^3] = 3$, $\dim[M_2(\mathbb{R})] = 4$, and $\dim[P_2(\mathbb{R})] = 3$. □

$\{E_{11}, E_{12}, E_{21}, E_{22}\}$

$\{1, x, x^2\}$

$$\boxed{\dim[\mathbb{R}^n] = n}, \quad \dim[M_{m\times n}(\mathbb{R})] = mn, \quad \dim[M_n(\mathbb{R})] = n^2, \quad \dim[P_n(\mathbb{R})] = n + 1.$$

$\{e_j\}_{j=1}^{n}$ $\{E_{ij}\}_{\substack{1\le i\le m \\ 1\le j\le n}}$ | $\{x^j : 0\le j\le n\}$

NEXT TIME

$\lvert S\rvert$	SPANNING	L.I.	BASIS
	$n = \dim V$		
$\lvert S\rvert < n$	NO	MAYBE	NO
$\lvert S\rvert = n$	EQUIV $\longleftrightarrow$	EQUIV $\longleftrightarrow$	EQUIV.
$\lvert S\rvert > n$	MAYBE	NO	NO