MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
T – 9:00 PM – 10:00 PM (ET)
F – 3:00 PM – 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1 - 11 ARE UPLOADED.

2. WW 06 - IS DUE WED (20th JULY AT 11:00 PM ET.
   WW 07 - IS DUE SUN (24th JULY) AT 11:00 PM ET.
   WW 08, 09 - IS DUE WED (27th JULY) AT 11:00 PM ET

3. MIDTERM 2 IS ON MONDAY (25th JULY) ~ SCHEDULER

   MORNING SLOT : 10:00 AM - 11:15 AM
   EVENING SLOT : 9:00 PM - 10:15 PM (UNLESS PROCTORED BY DIS. OFFICE)

4. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE !

§ 4.6 BASES & DIMENSION (CONT'D)

RECALL

DEFINITION 4.6.1

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is called a **basis**[5] for V if

(a) The vectors are linearly independent.

(b) The vectors span V.

RECALL

Remark More generally, consider the vector space of all $m \times n$ matrices with real entries, $M_{m\times n}(\mathbb{R})$. If we let E_{ij} denote the $m \times n$ matrix with value 1 in the (i, j)-position and zeros elsewhere, then one can show routinely that

$$\{E_{ij} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

is a basis for $M_{m\times n}(\mathbb{R})$, and it is the **standard basis** for $M_{m\times n}(\mathbb{R})$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Remark More generally, the reader can check that a basis for the vector space of all polynomials of degree n or less, $P_n(\mathbb{R})$, is

$$\{1, x, x^2, \ldots, x^n\}.$$

This is the **standard basis** for $P_n(\mathbb{R})$.

$\mathbb{R}^n \longrightarrow \{\vec{e}_1, \ldots, \vec{e}_n\}$

$e_j = \begin{cases} 1 & \text{at } j\text{th point} \\ 0 & \text{o.w.} \end{cases}$

RECALL : DIMENSION

DEFINITION 4.6.7

The **dimension** of a finite-dimensional vector space V, written $\dim[V]$, is the number of vectors in any basis for V. If V is the trivial vector space, $V = \{\mathbf{0}\}$, then we define its dimension to be zero.

Corollary 4.6.5 All bases in a finite-dimensional vector space V contain the same number of vectors.

$$\boxed{\dim[\mathbb{R}^n] = n, \quad \dim[M_{m\times n}(\mathbb{R})] = mn, \quad \dim[M_n(\mathbb{R})] = n^2, \quad \dim[P_n(\mathbb{R})] = n+1.}$$

n+1 elts.

$\{1, x, \dots, x^n\}$

RECALL

Theorem 4.6.4 If a finite-dimensional vector space has a basis consisting of n vectors, then any set of more than n vectors is linearly dependent.

$\rightarrow$ $|S| > \dim V \Rightarrow S$ IS L.D.

$\rightarrow$ S IS L.I. $\Rightarrow |S| \leq \dim V$

Corollary 4.6.6 If a finite-dimensional vector space V has a basis consisting of n vectors, then every spanning set for V must contain at least n vectors.

$\rightarrow$ $|S| < \dim V \Rightarrow S$ IS NOT SPANNING

$\rightarrow$ S IS SPANNING $\Rightarrow |S| \geq \dim V$

BASIS = L.I. + SPANNING

Theorem 4.6.10 If $\dim[V] = n$, then any set of n linearly independent vectors in V is a basis for V.

$|S| = n = \dim V$, S IS L.I. $\Rightarrow$ S IS A BASIS.

IT SUFFICES TO SHOW THAT S IS SPANNING.

$$S = \{v_1, \dots, v_n\}$$

$v \in V$. W.T.S. $v \in \operatorname{span}(S) = \operatorname{span}\{v_1, \dots, v_n\}$

WLOG $v \neq v_j$ FOR ANY j. $S' = \{v, v_1, \dots, v_n\}$

S' HAS $n+1$ ELEMENTS ($> \dim V$)

$\Rightarrow$ S' MUST BE L.D.

$$cv + c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

CLAIM : $c \neq 0$

$(\because$ IF $c = 0$, $c_1 v_1 + \cdots + c_n v_n = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0)$

$$v = \left(-\frac{c_1}{c}\right) v_1 + \left(\frac{-c_2}{c}\right) v_2 + \left(\frac{-c_3}{c}\right) v_3 + \cdots + \left(\frac{-c_n}{c}\right) v_n$$

$$\in \text{span}\{v_1, \ldots, v_n\}$$

$\Rightarrow$ $\{v_1, \ldots, v_n\}$ IS SPANNING (AND HENCE A BASIS).

Example 4.6.11 Verify that $S = \{\underbrace{1+x}_{p_1}, \overbrace{2-2x+x^2}^{p_2}, \underbrace{1+x^2}_{p_3}\}$ is a basis for $P_2(\mathbb{R})$.

$$|S| = 3 = \dim P_2(\mathbb{R})$$

IT SUFFICES TO CHECK THAT S IS L.I.

SUPPOSE $c_1 p_1 + c_2 p_2 + c_3 p_3 = 0$

$$\Rightarrow c_1(1+x) + c_2(2-2x+x^2) + c_3(1+x^2) = 0$$

$$\underbrace{(c_1+2c_2+c_3)}_{=0} + x\underbrace{(c_1-2c_2)}_{=0} + x^2\underbrace{(c_2+c_3)}_{=0} = 0$$

$$\left.\begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ c_1 - 2c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}\right\} \rightarrow A\vec{c} = \vec{0}$$

$$\Updownarrow$$

$$\vec{c} = \vec{0} \quad (\because A \text{ IS INV.})$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{A_{12}(-1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -4 & -1 \\ 1 & 1 \end{vmatrix} = -3 \neq 0$$

BASIS = L.I. + SPANNING

Theorem 4.6.12 If $\dim[V] = n$, then any set of n vectors in V that spans V is a basis for V.

$|S| = n = \dim V$

THEN, S SPANNING $\Rightarrow$ S IS A BASIS

NEED TO SHOW L.I.

SUPPOSE NOT.

$S' \subsetneq S$ s.t. S' IS ALSO SPANNING.

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \Rightarrow v_1 = \left(-\frac{c_2}{c_1}\right) v_2 + \cdots + \left(-\frac{c_n}{c_1}\right) v_n$

(WLOG, $c_1 \neq 0$)

CONTRADICTION, AS PREVIOUS THM. SAYS $|S'| < \dim V \Rightarrow S'$ IS NOT SPANNING

Corollary 4.6.13 If $\dim[V] = n$ and $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a set of n vectors in V, the following statements are equivalent:

1. S is a basis for V.
2. S is linearly independent.
3. S spans V.

I : S IS A BASIS $\Rightarrow$
① $|S| = \dim V$
② S IS SPANNING
③ S IS L.I.

II ANY 2 OF ①, ②, ③ ABOVE $\Rightarrow$ S IS A BASIS

Suppose that V is a vector space with $\dim[V] = n$,
and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of V.

S	$k < n$	$k > n$	$k = n$
is linearly independent?	Maybe	No (Theorem 4.6.4)	Maybe (Corollary 4.6.13)
spans V?	No (Corollary 4.6.6)	Maybe	Maybe (Corollary 4.6.13)
is a basis?	No (Corollary 4.6.5)	No (Corollary 4.6.5)	Maybe (Corollary 4.6.13)

Corollary 4.6.14 Let S be a subspace of a finite-dimensional vector space V. If $\dim[V] = n$, then

$$\dim[S] \leq n.$$

Furthermore, if $\dim[S] = n$, then $S = V$.

DIMENSION IS MONOTONIC

I.E. $S \subseteq V \Rightarrow \dim S \leq \dim V$

FURTHER $\dim S = \dim V \Leftrightarrow S = V$

PF. LET $\{v_1, \ldots, v_k\}$ BE A BASIS FOR S

$\Rightarrow \{v_1, \ldots, v_k\}$ IS L.I.

$\Rightarrow k \leq \dim V.$

$$\dim S \leq \dim V$$

PF OF EQUALITY

$$\dim S = \dim V = n$$

$\{v_1, \ldots, v_n\}$ BE A BASIS OF S.

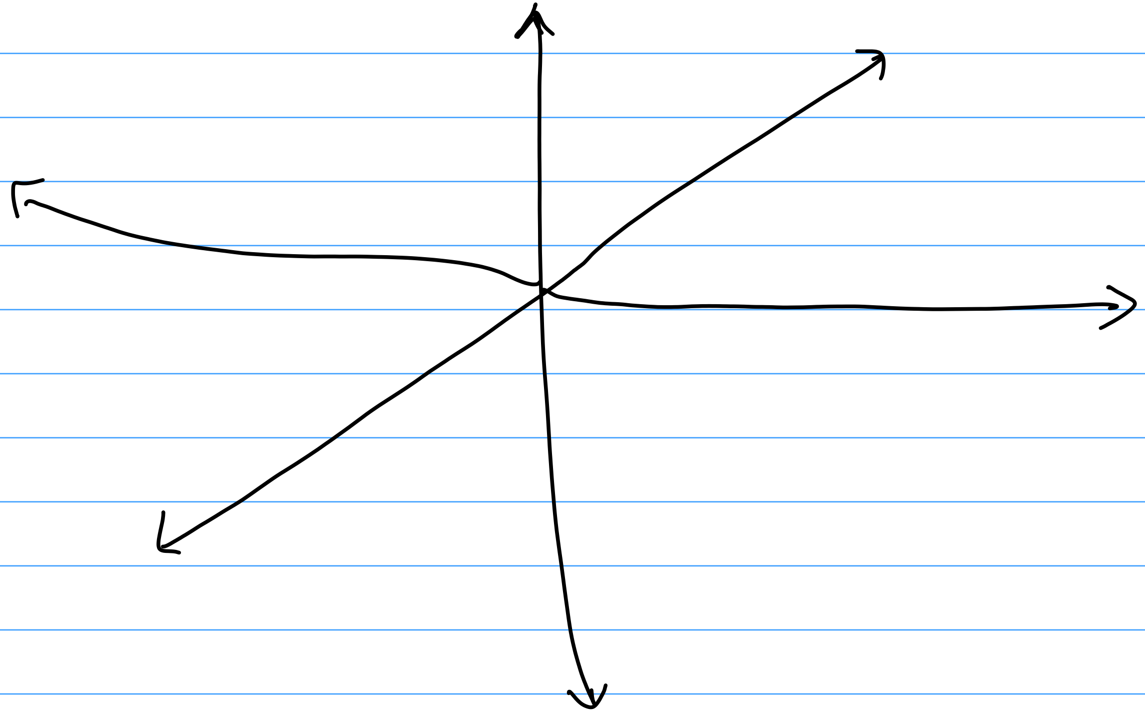
(1) $n = \dim V$

(2) $\{v_1, \ldots, v_n\}$ IS L.I. $\Bigg\} \Longrightarrow$ $\{v_1, \ldots, v_n\}$ SPANS V.

ON THE OTHER HAND, $\operatorname{span}\{v_1, \ldots, v_n\} \subseteq S \subseteq V$

$\Rightarrow S = V$

Example 4.6.15 Give a geometric description of the subspaces of $\mathbb{R}^3$ of dimensions 0, 1, 2, 3.



$\dim \mathbb{R}^3 = 3$

$S \leq \mathbb{R}^3$

$\dim S \leq 3$

$\dim S = 0, 1, 2, 3$

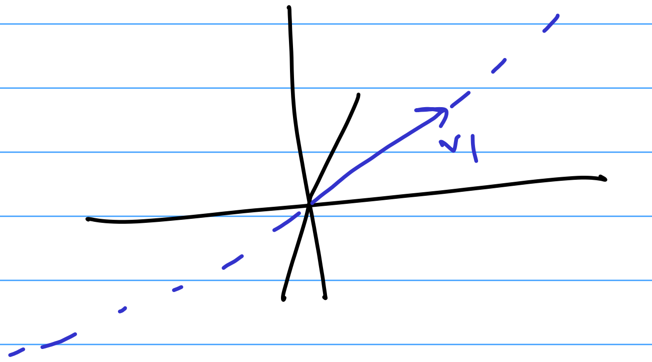
$\dim S = 3 \longrightarrow S = \mathbb{R}^3$

$\dim S = 0 \longleftrightarrow S = \{0\}$

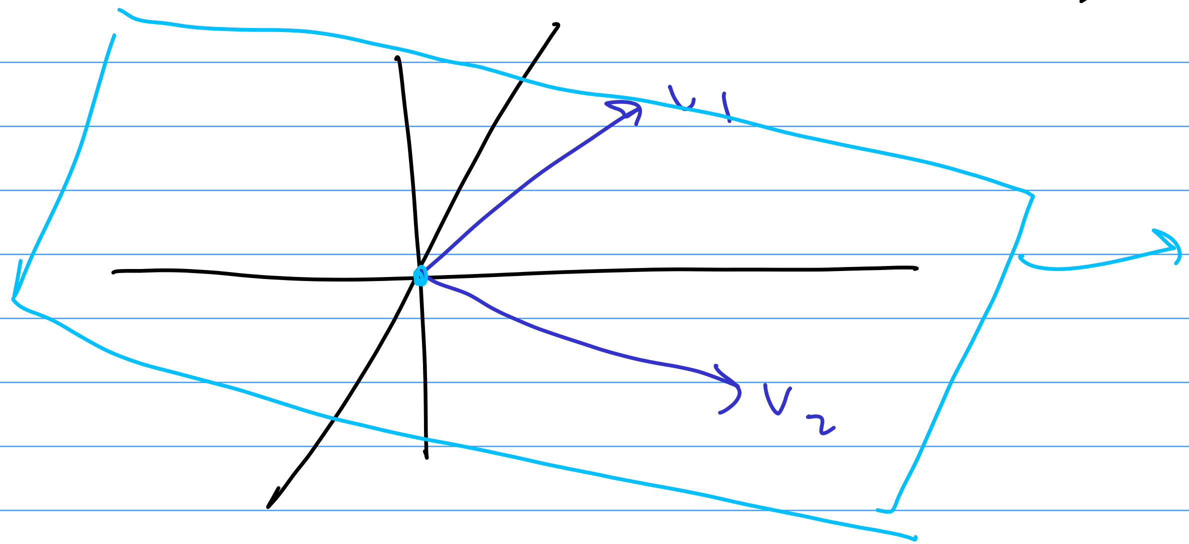
$\dim S = 1 \longleftrightarrow \{v_1\} \longleftrightarrow \{cv_1 : c \in \mathbb{R}^3\}$

$\downarrow$

LINE $\mathbb{R}^3$ IN



$\dim S = 2 \quad \leftrightarrow \quad \underset{(L.I.)}{\{v_1, v_2\}} \quad \leftrightarrow \quad \{c_1 v_1 + c_2 v_2 :$



Span $\{v_1, v_2\}$
$\parallel$
S

Example 4.6.16 Determine a basis for the subspace of $\mathbb{R}^3$ consisting of all solutions to the equation $x_1 + 2x_2 - x_3 = 0$.

STEP 1 : SOLVE THE SYSTEM OF EQN.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \end{array}\right] \longrightarrow \text{REDUCE ROW ECHELON FORM}$$

$$\left.\begin{array}{l} \text{RANK} = 1 \\ \text{\# OF VARIABLES} = 3 \end{array}\right\} \rightarrow \text{DEGREES OF FREEDOM} = 3 - 1 = 2$$

$\forall$ (FOR ALL)

$\exists$ (THERE EXISTS)

$$x_1 + 2x_2 - x_3 = 0$$

CAN PICK ANY 2 VARIABLE.

$$\left.\begin{aligned} x_2 &= s \\ x_3 &= t \end{aligned}\right\} \Rightarrow x_1 = -2x_2 + x_3 = t - 2s$$

$$\therefore (x_1, x_2, x_3) = (t-2s, s, t) \quad \forall s, t \in \mathbb{R}$$

$$= (t, 0, t) + (-2s, s, 0)$$

$$= t(1, 0, 1) + s(-2, 1, 0)$$

SOLUTION SET $= \text{span}\{ \underbrace{(1,0,1)}_{\vec{v}_1}, \underbrace{(-2,1,0)}_{\vec{v}_2} \}$

CLAIM: $\{\vec{v}_1, \vec{v}_2\}$ IS A BASIS (FOR THE SOLUTION SET)

$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$ $\Rightarrow$ $\{v_1, v_2\}$ IS L.I.

$c_1 (1,0,1) + c_2(-2,1,0) = \vec{0}$

$\Rightarrow (c_1 - 2c_2, c_2, c_1) = \vec{0}$

$\Rightarrow c_1 = 0$ $\quad c_1 - 2c_2 = 0$

$c_2 = 0$ $\quad \Rightarrow c_1 = c_2 = 0$

SKIP

Theorem 4.6.17 Let S be a subspace of a finite-dimensional vector space V. Any basis for S is part of a basis for V.

SKIP

Example 4.6.18 Let S denote the subspace of $M_2(\mathbb{R})$ consisting of all symmetric 2×2 matrices. Determine a basis for S, and find $\dim[S]$. Extend this basis for S to obtain a basis for $M_2(\mathbb{R})$.

BREAK TILL

10:00 AM

$$\boxed{\begin{array}{ll}\S\, 4.8 & \text{ROW - SPACE} \\ \& & \text{COLUMN - SPACE}\end{array}}$$

$A \to m \times n$ MATRIX, $A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}$

$(\vec{a}_j \in \mathbb{R}^n)$

$$\text{RowSpace}(A) = \text{Span}(\vec{a}_1, \dots, \vec{a}_m)$$

$$= \text{Span of Rows.}$$

$$(\subseteq \mathbb{R}^n)$$

Theorem 4.8.1 If A and B are row-equivalent matrices, then

$$\text{rowspace}(A) = \text{rowspace}(B).$$

$$A \xrightarrow{\text{EROs}} B$$

IN PARTICULAR, IF B IS IN (REDUCED) ROW ECHELON FORM.

Theorem 4.8.2 The set of nonzero row vectors in any row-echelon form of an $m \times n$ matrix A is a basis for rowspace(A).

$$A \xrightarrow{\text{EROs}} B \to \text{ROW-ECHELON FORM.}$$

$$\begin{array}{c} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \\ \\ \vdots \\ \vec{b}_k \end{array} \begin{bmatrix} 1 & * & * & \cdots & & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \end{bmatrix}$$

$$c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots c_k \vec{b}_k = 0$$

$m \times n$

$k \leq m$

$$b_j = [\,0 \;\cdots\; 0 \;\; 1 \;\; * \;\; * \;\cdots\; *\,]$$

$b_1 \longrightarrow$ LEADING 1.

$$c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_k \vec{b}_k = 0$$

$$\downarrow$$

$$c_1(1) + c_2(0) + \cdots + c_k(0) = 0$$

$\Rightarrow c_1 = 0$

$$c_2 b_2 + c_3 b_3 + \cdots + c_k b_k = 0$$

LEADING

1.

$$c_2(1) + c_3(0) + \cdots + c_k(0) = 0$$

$$\Rightarrow c_2 = 0$$

$$c_3 b_3 + \cdots + c_k b_k = 0$$

LEADING 1.

$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0$$

WE'VE PROVED: ROWS OF B ARE L.I.

$$\text{ROWSPACE}(B) = \operatorname{span}\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k, \vec{0}, \vec{0}\}$$
$$= \operatorname{span}\{\vec{b}_1, \ldots, \vec{b}_k\}$$

$$B = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vdots \\ \vec{b}_k \\ \vec{0} \\ \vec{0} \end{bmatrix}$$

$\Rightarrow$ $\{b_1, \ldots, b_k\}$ IS A BASIS OF ROWSPACE(B) = ROWSPACE(A)

Example 4.8.3 Determine a basis for the row space of

$$A = \begin{bmatrix} \boxed{1} & 4 & -1 & 2 & 3 & 5 \\ -2 & -7 & 5 & -5 & -6 & -9 \\ -2 & -6 & 8 & -6 & -6 & -8 \\ 1 & 5 & 2 & 1 & 3 & 6 \end{bmatrix}.$$

$A_{12}(2)$, $A_{13}(2)$, $A_{14}(-1)$

$$\begin{bmatrix} \boxed{1} & 4 & -1 & 2 & 3 & 5 \\ 0 & \boxed{1} & 3 & -1 & 0 & 1 \\ 0 & 2 & 6 & -10 & 0 & 2 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{bmatrix} \xrightarrow[A_{24}(-1)]{A_{21}(-4),\ A_{23}(-2)} \begin{bmatrix} \boxed{1} & 0 & -13 & 6 & 3 & 1 \\ 0 & \boxed{1} & 3 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c c}
\vec{b}_1 & \\
\vec{b}_2 & \\
\vec{0} & \\
\vec{0} &
\end{array}
\begin{bmatrix}
\boxed{1} & 0 & -13 & 6 & 3 & 1 \\
0 & \boxed{1} & 3 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \longrightarrow \text{REDUCED ROW ECHELON FORM}$$

$$\left\{ \begin{array}{l} b_1 = (1, 0, -13, 6, 3, 1) \\ b_2 = (0, 1, 3, -1, 0, 1) \end{array} \right.$$

BASIS. $\longrightarrow$ ROW SPACE (A) $= \text{span}\{\vec{b}_1, \vec{b}_2\}$, $\{b_1, b_2\}$ IS L.I

Example 4.8.4 Determine a basis for the subspace of $\mathbb{R}^4$ spanned by

$$\{\underbrace{(1,2,3,4)}_{v_1}, \underbrace{(4,5,6,7)}_{v_2}, \underbrace{(7,8,9,10)}_{v_3}\}.$$

$$A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix} \Rightarrow \text{ROWSPACE}(A) = \text{span}\{\vec{v}_1, \dots, \vec{v}_3\}$$

$$A \xrightarrow{\text{ERO's}} B \quad (\text{RREF})$$

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

$$\xrightarrow{A_{12}(-4),\ A_{13}(-7)} \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{-3} & -6 & -9 \\ 0 & -6 & -12 & -18 \end{bmatrix} \xrightarrow{M_2(-1/3)} \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & -6 & -12 & -18 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & -6 & -12 & -18 \end{bmatrix} \xrightarrow[A_{23}(6)]{A_{21}(-2)} \begin{array}{r} \vec{b}_1 \\ b_2 \\ \\ \end{array} \begin{bmatrix} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$b_1 = (1, 0, -1, 2), \quad b_2 = (0, 1, 2, 3)$$

$\{b_1, b_2\} \rightarrow$ BASIS FOR ROW SPACE (A) $\leftrightarrow$ BASIS FOR $\text{Span}(v_1, \ldots, v_3)$

COLUMN SPACE (A) = SPAN OF COLUMNS.

$A = [\vec{a}_1 \;\; \dots \;\; \vec{a}_n]$ $\to$ $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$

$\downarrow$

$m \times n$

$\vec{a}_j \in \mathbb{R}^m$

Example 4.8.5 For the matrix $A = \begin{bmatrix} 6 & 2 \\ -1 & 0 \\ 4 & -4 \end{bmatrix}$, we have

$$\text{colspace}(A) = \text{span}\left\{ \begin{bmatrix} 6 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} \right\},$$

$$A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$$

$\vec{v} \in$ Col Space (A)

$\Leftrightarrow \vec{v} \in$ Span $\{\vec{a}_1, \ldots, \vec{a}_n\}$

$$\vec{v} = c_1 \vec{a}_1 + \ldots + c_n \vec{a}_n$$

$$= \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A \vec{c}$$

$\Leftrightarrow \vec{v} = A\vec{c}$

Theorem 4.8.6 Let A be an $m \times n$ matrix. The set of column vectors of A corresponding to those column vectors containing leading ones in any row-echelon form of A is a basis for colspace(A).

(SEE PF IN BOOK)

SEE NEXT PAGE

Example 4.8.7 Determine a basis for colspace(A) if

$$A = \begin{array}{c} \begin{array}{ccccc} v_1 & & & v_2 & v_3 \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & -1 & \boxed{-2} & \boxed{-1} \\ 2 & 4 & -2 & -3 & -1 \\ 5 & 10 & -5 & -3 & -1 \\ -3 & -6 & 3 & 2 & 1 \end{bmatrix} \end{array}.$$

$\to \{v_1, v_2, v_3\}$ IS A BASIS OF COL. SPACE.

APPROACH 1 : LOOK A^T COMPUTE BASIS OF ROWSPACE(A^T)

APPROACH 2 : $A \xrightarrow{\text{EROs}} B$ (RREF)

$$\begin{bmatrix} \boxed{1} & 2 & -1 & -2 & -1 \\ 2 & 4 & -2 & -3 & -1 \\ 5 & 10 & -5 & -3 & -1 \\ -3 & -6 & 3 & 2 & 1 \end{bmatrix}$$

$$\Big\downarrow \begin{matrix} A_{12}(-2) \\ A_{13}(-5) \\ A_{14}(3) \end{matrix}$$

$$\begin{bmatrix} \boxed{1} & 2 & -1 & -2 & -1 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & -4 & -2 \end{bmatrix} \xrightarrow[\substack{A_{23}(-7) \\ A_{24}(4)}]{A_{21}(2)} \begin{bmatrix} \boxed{1} & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & \boxed{-3} \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & \boxed{-3} \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{M_3(-1/3)} \begin{bmatrix} \boxed{1} & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{A_{31}(-1),\ A_{32}(-1),\ A_{34}(-2)}$$

$$\begin{array}{c} \begin{bmatrix} \boxed{1} & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & & & \uparrow & \uparrow \\ \text{col 1} & & & \text{col 4} & \text{col 5} \end{array} \end{array}$$

NOTE : $A \xrightarrow{\text{EROs}} B$

DOES NOT IMPLY

THAT COLSPACE(A) = COLSPACE(B)

Summary: Let A be an $m \times n$ matrix. In order to determine a basis for rowspace(A) and a basis for colspace(A), we reduce A to row-echelon form.

> **1.** The row vectors containing the leading ones in the row-echelon form give a basis for rowspace(A) (a subspace of $\mathbb{R}^n$).
> **2.** The column vectors of A corresponding to the column vectors containing the leading ones in the row-echelon form give a basis for colspace(A) (a subspace of $\mathbb{R}^m$).

Since the number of vectors in a basis for rowspace(A) or in a basis for colspace(A) is equal to the number of leading ones in any row-echelon form of A, it follows that

$$\dim[\text{rowspace}(A)] = \dim[\text{colspace}(A)]. \quad = \text{RANK}(A)$$

<u>NOTE</u> : ROWSPACE $\neq$ COL. SPACE

§4.9 RANK-NULLITY THEOREM

$A \to m \times n$ MATRIX

$$\text{NULLSPACE}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$$

WE SHOWED $N(A)$ IS A SUBSPACE OF $\mathbb{R}^n$

$$\text{NULLITY}(A) = \dim N(A)$$

$A\vec{x} = \vec{0}$

Theorem 4.9.1 **(Rank-Nullity Theorem)**

For any $m \times n$ matrix A,

$$\underbrace{\text{rank}(A)}_{\text{dim (ROW SPACE)}} + \overbrace{\text{nullity}(A)}^{\text{dim (NULL-SPACE)}} = \underbrace{n}_{\dim \mathbb{R}^n}.$$

(PF NEXT TIME!)

RANK (SINCE ROWSPACE IS SPANNED BY 2nd ROW)

Example 4.9.2 If $A = \begin{bmatrix} 2 & -6 & -8 \\ -1 & 3 & 4 \\ 5 & -15 & -20 \\ -2 & 6 & 8 \end{bmatrix}$, find a basis for nullspace(A) and verify Theorem 4.9.1.

$A\vec{x} = \vec{0}$

RREF $\longrightarrow \begin{bmatrix} 1 & -3 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longleftrightarrow x_1 - 3x_2 - 4x_3 = 0$

DEGREES OF FREEDOM = 2

$x_2 = s$, $x_3 = t$

$x_1 = 3x_2 + 4x_3 = 3s + 4t$

$$(x_1, x_2, x_3) = (3s + 4t, s, t)$$

$$= s(3, 1, 0) + t(4, 0, 1)$$

$$\text{NULLSPACE}(A) = \{ A\vec{x} = \vec{0} \}$$

$$= \{ (3s + 4t, s, t) : s, t \in \mathbb{R} \}$$

$$= \text{span} \{ \underbrace{(3, 1, 0)}, \underbrace{(4, 0, 1)} \}$$

$\rightarrow$ L.I.

$$c_1(3, 1, 0) + c_2(4, 0, 1) = 0 \rightarrow c_1 = 0, \; c_2 = 0$$

$$\text{NULLITY}(A) = 2$$

NULLITY = 2

RANK = 1

RANK + NULLITY = 3 = # OF COLUMNS IN A.