MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
T — 9:00 PM – 10:00 PM (ET)
F — 3:00 PM – 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID: 979-4693-0650

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1 - 14 ARE UPLOADED.
2. WW 06 - WAS DUE WED (20th JULY) AT 11:00 PM ET.
   WW 07 - IS DUE SUN (24th JULY) AT 11:00 PM ET.
   WW 08, 09 - IS DUE WED (27th JULY) AT 11:00 PM ET
3. MIDTERM 2 IS ON MONDAY (25th JULY)
   ~> SCHEDULER
4. EXTRA OFFICE HOURS OVER THE WEEKEND.
5. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE !

§ 4.9 RANK-NULLITY THEOREM

RECALL : $A \to m \times n$ MATRIX

SPAN OF ROWS

SPAN OF COLUMNS

RANK (A) = dim (ROW SPACE (A)) = dim (COLUMN SPACE (A))

(= # OF LEADING 1s IN RREF)

NULLITY (A) = dim ($\underbrace{\text{NULL SPACE(A)}}$) $\{\vec{x} \in \mathbb{R}^n : A\vec{x} = 0\}$

Theorem 4.9.1 **(Rank-Nullity Theorem)**

For any $m \times n$ matrix A,

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n.$$

(RANK + NULLITY = # OF COLUMNS)

GENERATES THE ROW-SPACE

GENERATES THE COLUMN-SPACE

Example 4.9.2 If $A = \begin{bmatrix} 2 & -6 & -8 \\ -1 & 3 & 4 \\ 5 & -15 & -20 \\ -2 & 6 & 8 \end{bmatrix}$, find a basis for nullspace(A) and verify Theorem 4.9.1.

RANK 1 : $CA_{12}(3)$, $CA_{13}(4)$ $\rightsquigarrow \begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ 5 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$

(= dim (COL-SPACE) = 1)

$A\vec{x} = 0 \longrightarrow -x_1 + 3x_2 + 4x_3 = 0$

3 VAR, RANK 1 $\longrightarrow$ # OF DEGREES OF FREEDOM $= 3-1 = 2$

$x_2 = s$, $x_3 = t$, $x_1 = -3x_2 - 4x_3 = -3s - 4t$

$N(A) = \{ (-3s-4t, s, t) : s, t \in \mathbb{R} \}$

$(-3s-4t, s, t) = s(-3,1,0) + t(-4,0,1)$

$\Rightarrow N(A) = \text{SPAN}\{(-3,1,0), (-4,0,1)\}$ (L.I.)

$\therefore \text{NULLITY} = \dim N(A) = 2$

RANK + NULLITY = $1 + 2 = 3$ = # OF COLUMNS.

Pf of RANK-NULLITY THM

$\{N(A) \subseteq \mathbb{R}^n\}$

$k = \dim N(A)$

$\{v_1, \ldots, v_k\} \rightarrow$ BE A BASIS FOR $N(A)$

$\downarrow$

$\{v_1, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots v_n\} \rightarrow$ BASIS OF $\mathbb{R}^n$

(IDEA : $v_{k+1} \notin$ SPAN$(v_1, \ldots, v_k)$. IF $\text{span}\{v_1, \ldots, v_{k+1}\} = \mathbb{R}^n$ STOP.

ELSE : $v_{k+2} \notin$ SPAN $(v_1, \ldots, v_k, v_{k+1})$. ADD IT & REPEAT.

CLAIM: $\{Av_{k+1}, \dots, Av_n\}$ IS L.I.

Pf. $c_{k+1}(A\vec{v}_{k+1}) + c_{k+2}(A\vec{v}_{k+2}) + \dots + c_n(A\vec{v}_n) = 0$

$\Rightarrow A[c_{k+1}\vec{v}_{k+1} + \dots + c_n\vec{v}_n] = 0$

$\Rightarrow c_{k+1}\vec{v}_{k+1} + \dots + c_n\vec{v}_n \in N(A)$

$c_{k+1}\vec{v}_{k+1} + \dots + c_n\vec{v}_n = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$

$\Rightarrow -c_1\vec{v}_1 - c_2\vec{v}_2 - \dots - c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} + \dots + c_n\vec{v}_n = 0$

$\Rightarrow c_1 = c_2 = \dots = c_n = 0$

RECALL $\{v_1, \dots, v_n\}$ IS A BASES FOR $\mathbb{R}^n$

CLAIM 2 : $\underbrace{\{ A v_{k+1}, \ldots, A v_n \}}$ SPANS COL-SPACE

FOLLOWS FROM THE FACT THAT IF $\{\vec{v}_1, \ldots, \vec{v}_n\}$ SPANS $\mathbb{R}^n$, THEN

$\{A\vec{v}_1, \ldots, A\vec{v}_n\}$ SPANS COLUMN SPACE $(\mathbb{R}^n)$

$\vec{e}_1, \ldots, \vec{e}_n$

$A\vec{e}_j \longrightarrow$ jth COLUMN OF A

$$A = \begin{pmatrix} 2 & \boxed{3} & 1 & 4 \\ 5 & \boxed{6} & 7 & 8 \end{pmatrix} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{e}_2} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

2nd COLUMN

BUT $\quad Av_1 = Av_2 = \cdots = Av_k = 0 \quad (\because v_1, \dots, v_k \in N(A))$

$$\mathrm{SPAN}(Av_{k+1}, \dots, Av_n) = \mathrm{SPAN}(Av_1, \dots, Av_n)$$

$$= \text{COLUMN-SPACE}(A)$$

$$\dim \text{COL-SPACE} = n - k = n - \dim N(A)$$

	Description	Subspace of	Dimension
nullspace(A)	set of vectors $\mathbf{x}$ with $A\mathbf{x} = \mathbf{0}$	$\mathbb{R}^n$	nullity(A)
rowspace(A)	span of the row vectors of A	$\mathbb{R}^n$	rank(A)
colspace(A)	span of the column vectors of A	$\mathbb{R}^m$	rank(A)

EQUAL DIM

SUM = n.

$\rightarrow$ (# OF DEGREES OF FREEDOM) = n - RANK

Corollary 4.9.3 Let A be an $m \times n$ matrix, and consider the corresponding homogeneous linear system $\boxed{A\mathbf{x} = \mathbf{0}.} \rightarrow N(A)$

1. If $\text{rank}(A) = n$, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and so, $\text{nullspace}(A) = \{\mathbf{0}\}$.
2. If $\text{rank}(A) = r < n$, then $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions, all of which can be expressed in the form

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{n-r}\mathbf{x}_{n-r}, \tag{4.9.2}$$

where $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ is any linearly independent set of $n - r$ solutions to $A\mathbf{x} = \mathbf{0}$.

RANK $= n$ $\Rightarrow$ NULLITY $= n - n = 0$ $\Rightarrow$ $A\vec{x} = \vec{0}$ IF & ONLY IF $\vec{x} = \vec{0}$

OTOH, r = RANK $< n$ $\Rightarrow$ NULLITY $= n - r$ $\Rightarrow$ $A\vec{x} = \vec{0}$ $\vec{x} = c_1\vec{x}_1 + \cdots + c_{n-r}\vec{x}_{n-r}$

Theorem 4.9.4 Let A be an $m \times n$ matrix and consider the linear system $A\mathbf{x} = \mathbf{b}$.

1. If $\mathbf{b}$ is not in colspace(A), then the system is inconsistent.
2. If $\mathbf{b} \in$ colspace(A), then the system is consistent and has the following:
   - **(a)** a unique solution if and only if $\dim[\text{colspace}(A)] = n$.
   - **(b)** an infinite number of solutions if and only if $\dim[\text{colspace}(A)] < n$.

$$A\vec{x} = \vec{b}$$

CONSISTENT $\Leftrightarrow$ Rank A = Rank $[A \mid b]$

$\Leftrightarrow$ $b \in$ COLSPACE(A)

$$\vec{b} = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

$\Leftrightarrow$ $\vec{b} \in$ SPAN$(\vec{a}_1, \ldots, \vec{a}_n)$

$\Leftrightarrow$ $\vec{b} \in$ COLSPACE(A)

$$\mathrm{RANK}\,[A \mid b] \rightarrow \dim \mathrm{SPAN}(\vec{a}_1, \dots, \vec{a}_n, \vec{b})$$

$$= \dim \mathrm{SPAN}(\vec{a}_1, \dots, \vec{a}_n)$$

IF & ONLY IF

$$\vec{b} \in \mathrm{SPAN}(\vec{a}_1, \dots, \vec{a}_n)$$

$$= \mathrm{COLSPACE}(A)$$

Theorem 4.9.4 Let A be an $m \times n$ matrix and consider the linear system $A\mathbf{x} = \mathbf{b}$.

1. If $\mathbf{b}$ is not in colspace(A), then the system is inconsistent.
2. If $\mathbf{b} \in$ colspace(A), then the system is consistent and has the following:
   - **(a)** a unique solution if and only if dim[colspace(A)] $= n$.
   - **(b)** an infinite number of solutions if and only if dim[colspace(A)] $< n$.

2 (a) $\vec{b} \in$ COLSPACE(A), RANK $= n \Rightarrow$ NULLITY $= 0$

LET $\vec{x}_1$ & $\vec{x}_2$, s.t. $A\vec{x}_1 = \vec{b} = A\vec{x}_2$

$\Rightarrow A\vec{x}_1 - A\vec{x}_2 = 0 \Rightarrow A(\vec{x}_1 - \vec{x}_2) = 0$

$\Rightarrow \vec{x}_1 - \vec{x}_2 \in N(A) = \{0\}$

$\vec{x}_1 - \vec{x}_2 = \vec{0} \Rightarrow \vec{x}_1 = \vec{x}_2$

$r =$ RANK, NULLITY $= n - r$, $\vec{b} \in$ COLSPACE(A)

LET $\vec{x}_p$ BE A PARTICULAR SOL.

$\vec{x}$ BE A GEN. SOLN.

$$A\vec{x} = A\vec{x}_p = \vec{b}$$

$$\Rightarrow A(\vec{x} - \vec{x}_p) = 0 \Rightarrow \vec{x} - \vec{x}_p \in N(A)$$

$$\vec{x} - \vec{x}_p = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_{n-r}\vec{x}_{n-r}$$

WHERE $\{\vec{x}_1, \dots, \vec{x}_{n-r}\}$ IS A BASIS FOR $N(A)$

$Ax_j = 0$

Theorem 4.9.5 Let A be an $m \times n$ matrix. If $\operatorname{rank}(A) = r < n$ and $\mathbf{b} \in \operatorname{colspace}(A)$, then all solutions to $A\mathbf{x} = \mathbf{b}$ are of the form

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{n-r}\mathbf{x}_{n-r} + \mathbf{x}_p, \tag{4.9.3}$$

where $\mathbf{x}_p$ is any particular solution to $A\mathbf{x} = \mathbf{b}$, and $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ is a basis for nullspace(A).

Example 4.9.6 Let $A = \begin{bmatrix} 2 & -6 & -8 \\ -1 & 3 & 4 \\ 5 & -15 & -20 \\ -2 & 6 & 8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -18 \\ 9 \\ -45 \\ 18 \end{bmatrix}$. Verify that $\mathbf{x}_p = (5, 2, 2)$ is a particular solution to $A\mathbf{x} = \mathbf{b}$, and use Theorem 4.9.5 to determine the general solution to the system.

$$\underbrace{\begin{bmatrix} 2 & -6 & -8 \\ -1 & 3 & 4 \\ 5 & -15 & -20 \\ -2 & 6 & 8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}}_{\vec{x}_p} = \underbrace{\begin{bmatrix} -18 \\ 9 \\ -45 \\ 18 \end{bmatrix}}_{\vec{b}}$$

SOLVE: $A\vec{x} = 0 \longrightarrow \vec{x}_1, \dots, \vec{x}_{n-r}$ L.I. SOLN S.

BACK AT

10:02 AM

Q2. (a)

$$\det B = \begin{vmatrix} 1 & -1 & -1 & 1 \\ \boxed{1} & 2 & 2 & 1 \\ -2 & 0 & 4 & 1 \\ 0 & -2 & 3 & 4 \end{vmatrix}$$

→ COFACTOR EXPANSION

→ EROs

→ MIX

$$\Big\downarrow \; A_{21}(-1),\ A_{23}(2)$$

COFACTOR EXP.

$$\begin{vmatrix} 0 & -3 & -3 & 0 \\ 1 & 2 & 2 & 1 \\ 0 & 4 & 8 & 3 \\ 0 & -2 & 3 & 4 \end{vmatrix}$$

$$\det B = (-1)^{1+2} \begin{vmatrix} \boxed{-3} & \boxed{-3} & 0 \\ 4 & 8 & 3 \\ -2 & 3 & 4 \end{vmatrix}$$

$$\Big\downarrow \; CA_{12}(-1)$$

$$(-1) \begin{vmatrix} -3 & 0 & 0 \\ 4 & 4 & 3 \\ 2 & 5 & 4 \end{vmatrix}$$

$$\det B = (-1)\;(-1)^{1+1} \cdot (-3) \begin{vmatrix} 4 & 3 \\ 5 & 4 \end{vmatrix} = (3)\,(4\times 4 - 3\times 5)$$

$$= 3\,(1) = 3$$

4. (a) $p(x) \in V$. $p = b_1 + b_2 x + b_3 x^2$

DOES $p \in \text{SPAN}\{x, x^2, x+x^2, x-x^2\}$?

$$p(x) = c_1(x) + c_2(x^2) + c_3(x+x^2) + c_4(x-x^2)$$

$$\parallel$$

$$b_1 + b_2 x + b_3 x^2 = x(c_1 + c_3 + c_4) + x^2(c_2 + c_3 - c_4)$$

$b_1 = 0$, $b_3 = c_2 + c_3 - c_4$

$b_2 = c_1 + c_3 + c_4$

$$c_1 + c_3 + c_4 = b_2$$

$$c_2 + c_3 - c_4 = b_3$$

$$0 = b_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & b_2 \\ & 1 & 1 & -1 & b_3 \\ 0 & 0 & 0 & 0 & b_1 \end{array}\right]$$

$b_1 \neq 0 \Leftrightarrow$ NOT CONSISTENT

$p(x) = 1$ $\quad (b_1 = 1, b_2 = 6, b_3 = 0$

4. (b)

$$c_1 (0,3,1,4) + c_2 (2,3,5,7) + c_3 (1,2,3,4) = \vec{0}$$

$$(2c_2 + c_3,\ 3c_1 + 3c_2 + c_3,\ c_1 + 5c_2 + 3c_3,\ 4c_1 + 7c_2 + 4c_3) = \vec{0}$$

$$\begin{aligned} 2c_2 + c_3 &= 0 \\ 3c_1 + 3c_2 + c_3 &= 0 \\ c_1 + 5c_2 + 3c_3 &= 0 \\ 4c_1 + 7c_2 + 4c_3 &= 0 \end{aligned} \longrightarrow \begin{bmatrix} 0 & 2 & 1 \\ 3 & 3 & 1 \\ 1 & 5 & 3 \\ 4 & 7 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

$\rightarrow$ SOLVE SYSTEM

$\Rightarrow c_1 = c_2 = c_3 = 0$

SEE SUPPLEMENTAL NOTES FOR MORE DETAILS ON Q4.(b) & Q5 FROM THE EXAM.