Math 165: Linear Algebra w/ Diff. Equations

Sample Midterm 2

July 21st, 2022

NAME (please print legibly): ____________________

Your University ID Number: ____________________

- The exam will be 75 minutes long. You will get extra time in the end to upload the exam to Gradescope.
- There are 7 pages.
- ~~A formula sheet is provided.~~ (A LIST OF NOTATION)
- No calculators, phones, electronic devices, books, notes are allowed during the exam. The only materials you are allowed to use are are pen/pencil and paper. In particular, you are NOT allowed to take the exam on a tablet.
- You are allowed to use a phone or tablet to take photographs of your answer sheet once the exam is over. If you finish early, you must take permission before taking photographs. Once you start taking photographs, you are not allowed to write.
- **Show all work and justify all answers as much as possible.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

QUESTION	VALUE	SCORE
1	0	
2	25	
3	20	
4	20	
5	35	
TOTAL	100	

DIM = n+1

Notation

(WILL ADD DIMENSIONS)

- $P_n(\mathbb{R}) = \{p(x) : \deg p \leqslant n\}$ is the space of polynomials with real coefficients having degree at most n.
- $M_{m\times n}(\mathbb{R})$ is the space of all $m \times n$ matrices with real entries. → DIM = mn
- $M_n(\mathbb{R}) = M_{n\times n}(\mathbb{R})$ is the space of all square matrices of dimension n. → DIM = n^2

1. (0 points) Copy the following honesty pledge on to your answer sheet. Remember to sign and date it.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

2. (25 points)

Consider

ROW 2 = −2 × ROW 1

det A = 0

($A_{12}(2)$)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \\ 1 & \frac{1}{2022} & 2022 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 2 & 2 & 1 \\ -2 & 0 & 4 & 1 \\ 0 & -2 & 3 & 4 \end{bmatrix},$$

SEE NOTES (det (B) = 3)

$$C = \begin{bmatrix} -3 & 5 & 6 & -14 \\ 0 & 2 & 13 & -156 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

det (C) = (−3)(2)(−1/3)(5)
= 10

(a) Find the determinants of A, B, C. Which of A, B, C are invertible?

A → NOT INVERTIBLE

B, C → INVERTIBLE

(b) Find the determinant of ABC .

[det (AB) = det A det B]

det (ABC) = (det (A)) (det (BC))
= (0) · SOMETHING = 0

(c) Find the determinant of $C^T B$.

det ($C^T B$) = (det C^T) (det B)
= (det C) (det B)
= (10) (3) = 30

3. (20 points) For the following choices of S and V, determine whether S is a subspace of V.

(1) S IS CLOSED UNDER +

(2) S IS CLOSED UNDER IR - MUL

(a)

$$V = M_2(\mathbb{R}), S = \{A \in M_2(\mathbb{R}) : \det A = 1.\}$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, s.t. $ad - bc = 1$

$\det \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \neq 1 \Rightarrow \vec{0} \notin S$

NO ! (ZERO VECTOR TEST)

(b)

$$V = P_{\cancel{4}\,5}(\mathbb{R}), S = \{p(x) \in P_5(\mathbb{R}) : p(0) = p'(0) = 0\}.$$

$p, q \in S$

$c \in \mathbb{R}$

$(p+q)(0) = p(0) + q(0) = 0 + 0 = 0$

$(p+q)'(0) = p'(0) + q'(0) = 0 + 0 = 0$

$cp(0) = cp'(0) = 0 \Rightarrow p+q, cp \in S$

YES !

(c)

$$V = \mathbb{R}^3, S = \{(x, y, z) : xy = 0\}.$$

NO ! (NOT CLOSED UNDER +)

$(0,0,0) \to 0 \cdot 0 = 0, \vec{0} \in S$

$\vec{v}_1 = (1, 0, 0) \in S$

$\vec{v}_2 = (0, 1, 0) \in S$

$\vec{v}_1 + \vec{v}_2 = (1, 1, 0) \notin S$

$(x, y, z) \in S$

$(x', y', z') \in S$

$(x + x', y + y', z + z')$

$(x + x')(y + y') = \cancel{xy} + \cancel{x'y'} + xy' + x'y$

4. (20 points) For the following choices of finite sets S, determine whether S spans V, whether S is linearly independent in V, and whether S is a basis for V.

(a) $V = P_2(\mathbb{R})$ with $S = \{x, x^2, x + x^2, x - x^2\}$.

$\dim V = 2+1 = 3$, $|S| = 4$ $\Rightarrow$ S IS NOT L.I.
$\Rightarrow$ S IS NOT A BASIS

S IS SPANNING? $\rightarrow$ CHECK NOTES (ANS: NO!)

(b) $V = \mathbb{R}^4$ with $S = \{(0, 3, 1, 4), (2, 3, 5, 7), (1, 2, 3, 4)\}$.

$\dim V = 4$, $|S| = 3$ $\Rightarrow$ S IS NOT SPANNING
$\Rightarrow$ S IS NOT A BASIS

NEED TO CHECK: L.I. $\rightarrow$ (ANS: YES) $\rightarrow$ CHECK NOTES.

(c) $V = M_{2\times 3}(\mathbb{R})$ with

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\}$$

$\dim V = 2 \times 3 = 6$, $|S| = 6$

SUFFICES TO CHECK L.I.

$$c_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} + c_5 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 0$$

$$\left.\begin{aligned} c_1 + c_2 + c_3 + c_4 + c_5 &= 0 \\ c_2 + c_3 + c_4 + c_5 &= 0 \\ c_3 + c_4 + c_5 &= 0 \\ c_4 + c_5 &= 0 \\ c_5 &= 0 \end{aligned}\right\} \Rightarrow$$

$c_1 = c_2 = \dots = c_5 = 0$ (BACK-SUBS.)

$\therefore$ L.I.

$\Rightarrow$ ALSO SPANNING

$\Rightarrow$ BASIS.

5. (35 points) Consider the matrix

$$\overbrace{\begin{bmatrix} 4 & 3 & 2 & -1 \\ 5 & 4 & 3 & -1 \\ -2 & -2 & -1 & 2 \\ 11 & 6 & 4 & 1 \end{bmatrix}}^{A} \xrightarrow{\text{EROs}} \overbrace{\begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}^{B}$$

(a) Find a basis for the row-space, and hence compute the dimension of the row-space.

BASIS $= \{(1,0,0,1), (0,1,0,-3), (0,0,1,2)\}$

DIM (ROW-SPACE) $= 3$

(b) Find a basis for the column-space, and hence compute the dimension of the column-space.

BASIS $= \{(4, 5, -2, 11), (3, 4, -2, 6), (2, 3, -1, 4)\}$

DIM (COL-SPACE) $= 3$

(c) Find a basis for the null-space, and hence compute the dimension of the null-space.

(# OF VAR = 4, RANK = 3)

$A\vec{x} = 0 \iff B\vec{x} = 0$

$x_1 + x_4 = 0$

$x_2 - 3x_4 = 0$

$x_3 + 2x_4 = 0$

CHOOSE, $x_4 = t$

$\vec{x} = (-t, 3t, -2t, t) = t(-1, 3, -2, 1) \Rightarrow N(A) = \text{SPAN}\{(-1,3,-2,1)\}$

DIM (NULL-SPACE) $= 1$

(d) Do your answers in the previous parts match your expectation? Explain.

EXPECT:

✓ DIM (ROWSPACE) = DIM (COL-SPACE) = RANK (A)

✓ $\text{RANK}(A) + \underbrace{\text{NULLITY}(A)}_{\text{DIM (NULL-SPACE)}} = \#$ OF COLS.

$3 + 1 = 4$