MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
T – 9:00 PM – 10:00 PM (ET)
F – 3:00 PM – 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEBPAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1-15 ARE UPLOADED.

2. WW 08, 09 - IS DUE WED (27th JULY) AT 11:00 PM ET
   WW 10, 11 - IS DUE MON (1st AUG) AT 11:00 PM ET

3. HARD WEBWORK DEADLINE : FRIDAY, 5th AUG

4. MIDTERM 2 IS TODAY. [10 AM OR 9 PM]

5. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE!

$\boxed{\text{§ 6.1 DEFN. OF A LINEAR TRANSFORMATION}}$

DEFINITION 6.1.1

Let V and W be vector spaces. A **mapping** T from V into W is a rule that assigns to each vector $\mathbf{v}$ in V precisely one vector $\mathbf{w} = T(\mathbf{v})$ in W. We denote such a mapping by $T : V \to W$.

$v \in V \rightsquigarrow T(v) \in W$

$P_n(\mathbb{R}) \to$ POLYS OF deg $\leq n$

$M_n(\mathbb{R}) \to n \times n$ SQUARE MATRIX

Example 6.1.2 The following are examples of mappings between vector spaces:

1. $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $T(A) = A^T$.
2. $T : M_n(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = \det(A)$.
3. $T : P_1(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(a_0 + a_1 x) = 2a_0 + a_1 + (a_0 + 3a_1)x + 4a_1 x^2$.
4. $T : C^0[a, b] \to \mathbb{R}$ defined by $T(f) = \int_a^b f(x)\, dx$. □

$C^0[a,b] \to$ SPACE OF CONTINUOUS FUNCTIONS ON $[a,b]$

$\{ f : [a,b] \to \mathbb{R} : f \text{ CONT.} \}$

$$V \longrightarrow (\mathbb{R}, \overset{+}{\cdot})$$

DEFINITION 6.1.3

Let V and W be vector spaces.[1] A mapping $T : V \to W$ is called a **linear transformation** from V to W if it satisfies the following properties:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for *all* $\mathbf{u}, \mathbf{v} \in V$.
2. $T(c\,\mathbf{v}) = c\,T(\mathbf{v})$ for *all* $\mathbf{v} \in V$ and *all* scalars c.

} RESPECTING THE VECTOR SPACE STRUCTURE

We refer to these properties as the **linearity properties**. The vector space V is called the $\boxed{\textbf{domain of } T}$, while the vector space W is called the $\boxed{\textbf{codomain of } T}$.

NOTE: $T(\vec{u} \underbrace{+}_{\text{IN } V} \vec{v}) = T(u) \underbrace{+}_{\text{IN } W} T(v)$

Example 6.1.2 The following are examples of mappings between vector spaces:

1. $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $T(A) = A^T$. } → LINEAR
2. $T : M_n(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = \det(A)$. } → NOT LINEAR
3. $T : P_1(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(a_0 + a_1x) = 2a_0 + a_1 + (a_0 + 3a_1)x + 4a_1x^2$. } → LINEAR
4. $T : C^0[a, b] \to \mathbb{R}$ defined by $T(f) = \int_a^b f(x)\,dx$. } → LINEAR □

$A, B \in M_n(\mathbb{R}) \quad , \quad c \in \mathbb{R}$

(RESPECTS ADD.) $\quad T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$

(RESP. SCALAR MULT.) $\quad T(cA) = [cA]^T = cA^T = cT(A)$

$A \in M_n(\mathbb{R}) \qquad c \in \mathbb{R}$

$T(cA) = \det(cA) = c^n \det A = c^n T(A)$

$n=2$, $\quad T(2A) = 4T(A) \neq 2T(A)$

$T : C^0[a,b] \to \mathbb{R} \qquad T(f) = \int_a^b f(x)\,dx$

$f, g \in C^0[a,b]$, $\quad c \in \mathbb{R}$

RESP. +

$T(f+g) = \int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$= T(f) + T(g)$

RESP.

$\mathbb{R}$.

$$T(cf) = \int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

$$= c\,T(f)$$

CONT. FUNCT.

Example 6.1.4 Define $T : C^1(I) \to C^0(I)$ by $T(f) = f'$. Verify that T is a linear transformation.

DIFF.
FUNCTIONS

$$T(f+g) = (f+g)' = f' + g' = T(f) + T(g)$$

$$T(cf) = (cf)' = cf' = cT(f)$$

Example 6.1.5 Define $T : C^2(I) \to C^0(I)$ by $T(y) = y'' + y$. Verify that T is a linear transformation.

TWICE - DIFF.

$$T(y_1 + y_2) = (y_1 + y_2)'' + (y_1 + y_2)$$
$$= y_1'' + y_2'' + y_1 + y_2$$
$$= (y_1'' + y_1) + (y_2'' + y_2)$$
$$= T(y_1) + T(y_2)$$

$$T(cy) = (cy)'' + cy = cy'' + cy = c(y'' + y)$$
$$= cT(y)$$

$M_{2\times 3} \rightarrow$ 2×3 MATRICES.

Example 6.1.6 Define $T : M_{23}(\mathbb{R}) \rightarrow M_2(\mathbb{R})$ by

$$T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} c+3f & -b \\ -b & 4a-3d \end{bmatrix}.$$

Verify that T is a linear transformation.

$$T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} + \begin{bmatrix} a' & b' & c' \\ d' & e' & f' \end{bmatrix}\right) = T\left(\begin{bmatrix} \cdots \end{bmatrix}\right)$$

$$+\, T\left(\begin{bmatrix} \cdots \end{bmatrix}\right)$$

A SINGLE CRITERION.

Theorem 6.1.7 A mapping $T : V \to W$ is a linear transformation if and only if

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2),$$

for all $\mathbf{v}_1, \mathbf{v}_2$ in V and all scalars c_1, c_2.

Pf

IF T IS LINEAR

$$T(c_1\vec{v}_1 + c_2\vec{v}_2) = T(c_1\vec{v}_1) + T(c_2 v_2)$$

$$= c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2)$$

If $T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2)$

$c_1 = c_2 = 1$; $T(v_1 + v_2) = 1 \cdot T(v_1) + 1 \cdot T(v_2) = T(v_1) + T(v_2)$

$c_2 = 0, \quad c_1 = c, \quad v_1 = v$

$$T(cv + 0 \cdot v_2) = c\,T(v) + 0 \cdot T(v_2)$$

$$\Rightarrow T(cv) = c\,T(v)$$

$\Rightarrow$ T IS LINEAR.

Example 6.1.8 Define $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ via

$$T(p(x)) = (p(2), p'(4)).$$

deg ≤ 2

Verify that T is a linear transformation.

$p_1, p_2 \in P_2(\mathbb{R}) \qquad c_1, c_2 \in \mathbb{R}$

$$T(c_1 p_1 + c_2 p_2) = \left((c_1 p_1 + c_2 p_2)(2), (c_1 p_1 + c_2 p_2)'(4) \right)$$

$$(c_1 p_1 + c_2 p_2)(2) = c_1 p_1(2) + c_2 p_2(2)$$

$$(c_1 p_1 + c_2 p_2)'(4) = c_1 p_1'(4) + c_2 p_2'(4)$$

$$= (c_1 p_1(2), c_1 p_1'(4)) + (c_2 p_2(2), c_2 p_2'(4))$$

$$= c_1 T(p_1) + c_2 T(p_2)$$

Example 6.1.9 If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that

$$T(1,0,0) = (7,-2), \qquad T(0,1,0) = (1,5), \qquad T(0,0,1) = (0,-8),$$

then we can compute

$$T(4,3,2)$$

$$(4,3,2) = 4(1,0,0) + 3(0,1,0) + 2(0,0,1)$$

$$T(4,3,2) = T\left[4(1,0,0) + 3(0,1,0) + 2(0,0,1)\right]$$

$$= T[4(1,0,0)] + T[3(0,1,0)] + T[2(0,0,1)]$$

$$= 4T(1,0,0) + 3T(0,1,0) + 2T(0,0,1)$$

$$= 4(7,-2) + 3(1,5) + 2(0,-8) = (31, -1)$$

$1, x, x^2$

Example 6.1.10 Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation satisfying

$$T(1) = 2 - 3x, \qquad T(x) = 2x + 5x^2, \qquad T(x^2) = 3 - x + x^2.$$

For an arbitrary vector $p(x) = a_0 + a_1 x + a_2 x^2$ in $P_2(\mathbb{R})$, determine $T(p(x))$.

$$\begin{aligned}
T(a_0 \cdot 1 + a_1 x + a_2 x^2) &= T(a_0 \cdot 1) + T(a_1 x) + T(a_2 x^2) \\
&= a_0 T(1) + a_1 T(x) + a_2 T(x^2) \\
&= a_0(2 - 3x) + a_1(2x + 5x^2) \\
&\qquad + a_2(3 - x + x^2)
\end{aligned}$$

$$\boxed{\begin{aligned}
T(a_0 + a_1 x + a_2 x^2) &= (2a_0 + 3a_2) + x(2a_1 - 3a_0 - a_2) \\
&\qquad + x^2(5a_1 + a_2)
\end{aligned}}$$

Theorem 6.1.11 Let $T : V \to W$ be a linear transformation. Then

1. $T(\mathbf{0}_V) = \mathbf{0}_W$, ✓
2. $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v} \in V$.

Pf of ①: $0 \cdot 0_V = 0_V$

$$T(0_V) = T(0 \cdot 0_V)$$

$$= 0 \cdot T(0_V)$$

$$= 0_W$$

$\boxed{\text{Pf of } ②}$

$v \in V,$

$w = -v \Rightarrow v + w = 0_V$

$$T(v+w) = T(0_V) = 0_W \quad [\text{BY } ①]$$

OTOH, $T(v+w) = T(v) + T(w) = T(v) + T(-v)$

$$T(v) + T(-v) = 0_W$$

$$\Rightarrow T(-v) = -T(v)$$

[BY UNIQUENESS OF ADDITIVE INVERSE]

Example 6.1.12 Define $T : \mathbb{R}^2 \to \mathbb{R}^4$ as follows: If $\mathbf{x} = (x_1, x_2)$, then

$$T(\mathbf{x}) = (2x_1 + x_2, 3x_1 - x_2, -5x_1 + 3x_2, -4x_2).$$

Verify that T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

$$T(c_1\vec{x} + c_2\vec{y}) = c_1 T(\vec{x}) + c_2 T(\vec{y})$$

$$\vec{x} = (x_1, x_2)$$

$$\vec{y} = (y_1, y_2)$$

Theorem 6.1.13 Let A be an $m \times n$ real matrix, and define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

$\vec{x} \in \mathbb{R}^n \to$ COLUMN n-VECTOR ($n \times 1$ MATRIX)

$A \to$ ($m \times n$ MATRIX)

$A\vec{x} \in \mathbb{R}^m \to$ ($m \times 1$ MATRIX) COLUMN m-VECTOR

$\vec{x}, \vec{y} \in \mathbb{R}^n$

$$T(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = \underbrace{A\vec{x}}_{} + A\vec{y} = T(\vec{x}) + T(\vec{y})$$

($\because$ MATRIX MULT. IS DISTRIBUTIVE)

(RESPECTS +)

$$T(c\vec{x}) = A(c\vec{x}) = c(A\vec{x}) = cT(\vec{x})$$

(RESPECTS SCALAR MULTIP.)

$\rightarrow (x_1, x_2) \rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Example 6.1.14 Determine the matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ if

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \\ -5 & 3 \\ 0 & -4 \end{bmatrix}.$$

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 2 & 1 \\ 3 & -1 \\ -5 & 3 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ 3x_1 - x_2 \\ -5x_1 + 3x_2 \\ -4x_2 \end{bmatrix}$$

$$T(\vec{x}) = (2x_1 + x_2, 3x_1 - x_2, -5x_1 + 3x_2, -4x_2)$$

$T(\vec{x}) = (2x_1 + x_2, 3x_1 - x_2, -5x_1 + 3x_2, -4x_2)$

Example 6.1.12 Define $T : \mathbb{R}^2 \to \mathbb{R}^4$ as follows: If $\mathbf{x} = (x_1, x_2)$, then

$$T(\mathbf{x}) = (2x_1 + x_2, 3x_1 - x_2, -5x_1 + 3x_2, -4x_2).$$

Verify that T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

Theorem 6.1.15 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is described by the matrix transformation

$$T(\mathbf{x}) = A\mathbf{x},$$

where A is the $m \times n$ matrix

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)]$$

and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the standard basis vectors in $\mathbb{R}^n$.

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\vec{x} \in \mathbb{R}^n$

$\vec{x} = (x_1, x_2, \ldots, x_n) = x_1(1, 0, \ldots, 0) + x_2(0, 1, 0, 0, \ldots) + \cdots + x_n(0, \ldots, 1)$

$= x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n$

$$T(\vec{x}) = T(x_1 \vec{e}_1 + \dots + x_n \vec{e}_n)$$

$$= T(x_1 \vec{e}_1) + T(x_2 \vec{e}_2) + \dots + T(x_n \vec{e}_n)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \dots + x_n T(\vec{e}_n)$$

$$= \underbrace{\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_n) \end{bmatrix}}_{:= A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x \text{ (IN COLUMN FORM)}}$$

$$T(\vec{x}) = A \vec{x}$$

DEFINITION 6.1.16

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the $m \times n$ matrix

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)]$$

is called the **matrix of** T.

UPSHOT :

$m \times n$ MATRIX $\Longleftrightarrow$ $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ LINEAR