MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
T – 9:00 PM – 10:00 PM (ET)
F – 3:00 PM – 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1-16 ARE UPLOADED.

2. WW 08, 09 - IS DUE WED (27th JULY) AT 11:00 PM ET
   WW 10, 11 - IS DUE MON (1st AUG) AT 11:00 PM ET

3. HARD WEBWORK DEADLINE : FRIDAY, 5th AUG

4. MIDTERM 2 SOLNS. ARE ONLINE.

5. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE!

§ 6.1 DEFN. OF A LINEAR TRANSFORMATION (CONTD.)

RECALL:

Theorem 6.1.13 Let A be an $m \times n$ real matrix, and define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

Theorem 6.1.15 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is described by the matrix transformation

$$T(\mathbf{x}) = A\mathbf{x},$$

where A is the $m \times n$ matrix

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)]$$

and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the standard basis vectors in $\mathbb{R}^n$.

UPSHOT :

$m \times n$ MATRIX $\iff$ $T: \mathbb{R}^n \to \mathbb{R}^m$
LINEAR

DEFINITION 6.1.16

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the $m \times n$ matrix

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)]$$

is called the **matrix of** T.

CHECK

Example 6.1.17 Determine the matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T(x_1, x_2, x_3) = (-x_1 + 3x_3, -2x_3, 2x_1 + 5x_2 - 9x_3, -7x_1 + 5x_2). \tag{6.1.3}$$

$\mathbb{R}^3 \longrightarrow \vec{e}_1, \vec{e}_2, \vec{e}_3$

$\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$

$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix}$

$T(\vec{e}_1) = T(1,0,0) = (-1, 0, 2, -7)$

$T(\vec{e}_2) = T(0,1,0) = (0, 0, 5, 5)$

$T(\vec{e}_3) = T(0,0,1) = (3, -2, -9, 0)$

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & -2 \\ 2 & 5 & -9 \\ -7 & 5 & 0 \end{bmatrix}$$

CHECK

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\vec{x} = T(\vec{x})$$

§ 6.3 KERNEL & RANGE

$Ker(T) \subseteq V$
[SUBSPACE]

DEFINITION 6.3.1

Let $T : V \to W$ be a linear transformation. The set of *all* vectors $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{0}$ is called the **kernel** of T and is denoted $\mathrm{Ker}(T)$. Thus,

$\in W$

$$\mathrm{Ker}(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}.$$

e.g. $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\vec{x}) = A\vec{x}$

$$Ker(T) = \{ v \in V : T(\vec{v}) = 0 \}$$

$$= \{ v \in V : A\vec{v} = 0 \} = \text{NULL-SPACE}(A)$$

Example 6.3.2 Determine Ker(T) for the linear transformation $T : C^2(I) \to C^0(I)$ in Example 6.1.5 defined by $T(y) = y'' + y$.

$$T : C^2(I) \longrightarrow C^0(I)$$

$$T(y) = y'' + y$$

$$\text{Ker } T = \{ y \in C^2(I) : T(y) = 0 \}$$

$$= \{ y \in C^2(I) : \underbrace{y'' + y = 0}_{\text{2nd ORDER LINEAR ODE.}} \}$$

$$= \text{span}(\cos x, \sin x)$$

CAN SHOW: $y = c_1 \cos x + c_2 \sin x$

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation with matrix A then $\mathrm{Ker}(T)$ is the solution set to the homogeneous linear system $\underbrace{A\mathbf{x} = \mathbf{0}}_{\text{NULLSPACE}}$.

$\therefore$ KERNEL SIMULTANEOUSLY GENERALIZES BOTH PROBLEMS

DEFINITION 6.3.3

The **range** of the linear transformation $T : V \to W$ is the subset of W consisting of all transformed vectors from V. We denote the range of T by $\operatorname{Rng}(T)$. Thus,

$$\operatorname{Rng}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}.$$

e.g. $T(x_1, x_2) = (x_1, x_2, 0)$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\text{Range}(T) = \{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$

$= \{\vec{x} \in \mathbb{R}^3 : x_3 = 0\}$

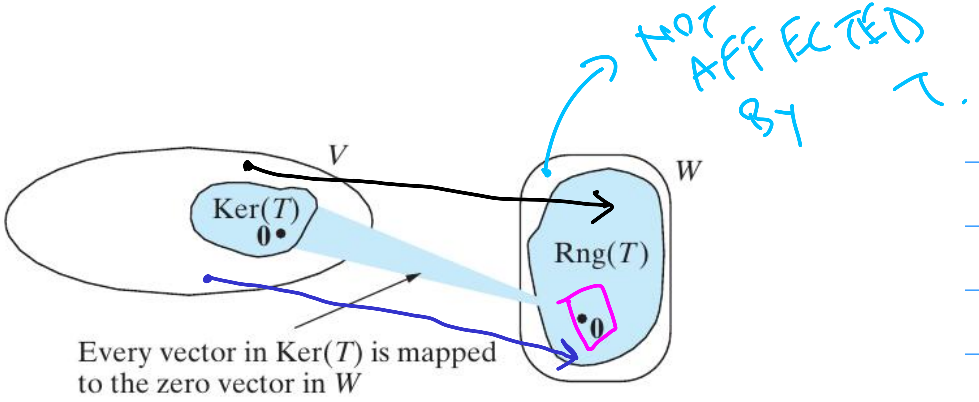

Figure 6.3.1: Schematic representation of the kernel and range of a linear transformation.

$T(0) = 0 \in Rng(T)$

If $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ denotes the matrix of T, then

$$\begin{aligned}\mathrm{Rng}(T) &= \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} \\ &= \{x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n : x_1, x_2, \ldots, x_n \in \mathbb{R}\} \\ &= \mathrm{colspace}(A).\end{aligned}$$

$T(\vec{x}) = A\vec{x}$ $\quad \{ T : \mathbb{R}^n \to \mathbb{R}^m, \; A \to m \times n \text{ MATRIX} \}$

$$T(\vec{x}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

COLUMN DECOMPOSITION

$$\in \mathrm{Span}(\vec{a}_1, \ldots, \vec{a}_n)$$

$$= \mathrm{COLSPACE}(A)$$

$$\mathrm{Rng}(T) = \mathrm{Span}(\vec{a}_1, \ldots, \vec{a}_n) = \mathrm{COLSPACE}(A)$$

Example 6.3.4 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation with matrix $A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \end{bmatrix}$. Determine $\text{Ker}(T)$ and $\text{Rng}(T)$.

USE PREV. METHODS.

$$\text{Ker}(T) = \text{NULL SPACE}(A) = \{\vec{x} : \boxed{A\vec{x} = 0}\}$$

$$\text{Rng}(T) = \text{COLSPACE}(A) = \text{span}\left((-1,2), \underbrace{(-2,4)}_{\times 2}, \underbrace{(5,-10)}_{\times(-5)}\right)$$

$$= \text{span}\left((-1,2)\right)$$

To summarize, any matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with $m \times n$ matrix A has natural subspaces

$$\boxed{\begin{array}{rll} \mathrm{Ker}(T) &= \mathrm{nullspace}(A) & \quad (\text{subspace of } \mathbb{R}^n) \\ \mathrm{Rng}(T) &= \mathrm{colspace}(A) & \quad (\text{subspace of } \mathbb{R}^m) \end{array}}$$

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\qquad\qquad$ $T : V \longrightarrow W$

COL-SPACE $\qquad\qquad$ RANGE

NULL-SPACE $\qquad\qquad$ KERNEL

$v \in V$ s.t. $T(v) = 0$

Theorem 6.3.5 If $T : V \to W$ is a linear transformation, then

1. Ker(T) is a subspace of V.
2. Rng(T) is a subspace of W.

Pf of ①: $\vec{v}_1, \vec{v}_2 \in$ Ker(T) $\quad c_1, c_2 \in \mathbb{R}$

$$T(c_1\vec{v}_1 + c_2\vec{v}_2) = T(c_1\vec{v}_1) + T(c_2\vec{v}_2)$$

$$= c_1 \underbrace{T(\vec{v}_1)}_{=0} + c_2 \underbrace{T(\vec{v}_2)}_{=0}$$

$$= c_1 0 + c_2 0 = 0$$

$\Rightarrow c_1\vec{v}_1 + c_2\vec{v}_2 \in$ Ker(T)

Pf of ②: $\vec{w}_1, \vec{w}_2 \in \text{Rng}(T)$; $c_1, c_2 \in \mathbb{R}$

$$\left.\begin{array}{l} \vec{w}_1 = T(\vec{v}_1) \\ \vec{w}_2 = T(\vec{v}_2) \end{array}\right\} \exists\, v_1, v_2 \in V$$

$$c_1 \vec{w}_1 + c_2 \vec{w}_2 = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2)$$

$$= T(\underbrace{c_1 \vec{v}_1 + c_2 \vec{v}_2}_{\in V}) \in \text{Rng}(T)$$

$\Rightarrow$ $\text{Ker}(T) \subseteq V$, $\text{Rng}(T) \subseteq W$ ARE SUBSPACES.

Example 6.3.6 Find Ker(S), Rng(S), and their dimensions for the linear transformation $S : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by

$$S(A) = A - A^T.$$

$$\mathrm{Ker}(S) = \{ A \in M_2(\mathbb{R}) : S(A) = A - A^T = 0 \}$$

$$= \{ A \in M_2(\mathbb{R}) : \underbrace{A = A^T}_{\text{SYMMETRIC}} \}$$

$$\mathrm{Rng}(S) = \{ S(A) \in M_2(\mathbb{R}) : A \in M_2(\mathbb{R}) \}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow S(A) = A - A^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$S(A) = A - A^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$= \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}$$

$$= (b-c) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\in \operatorname{span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

$$\operatorname{Rng}(S) = \operatorname{span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \longrightarrow \text{SKEW-SYMMETRIC}$$

SKEW-SYMMETRIC $= \{ A \in M_2(\mathbb{R}) : A^T = -A \}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$a = -a$, $b = -c$, $c = b$, $d = -d$

$\Rightarrow a = d = 0$, $b = -c$

$$= \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} : b \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

$$\dim(\ker(T)) = 3$$

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \in \operatorname{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\dim(\operatorname{Rng}(T)) = 1$$

$$\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \in \operatorname{Span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

$$\text{SUM} = 1 + 3 = 4 = \dim M_2(\mathbb{R})$$

$$(2 \times 2)$$

RANK + NULLITY = # OF COLUMNS.

?

Theorem 6.3.8 **(General Rank-Nullity Theorem)**

If $T : V \to W$ is a linear transformation and V is finite-dimensional, then

$$\dim[\mathrm{Ker}(T)] + \dim[\mathrm{Rng}(T)] = \dim[V].$$

RANK $\longrightarrow$ dim (COL-SPACE)

NULLITY $\longrightarrow$ dim (NULL-SPACE)

# of COLUMNS = n = dim (DOMAIN)

$T : \mathbb{R}^n \to \mathbb{R}^m \rightsquigarrow$ A IS $m \times n$

$p(x) = ax^2 + bx + c$

Example 6.3.9 Let $T : \underbrace{P_2(\mathbb{R})}_{\nearrow 2+1=3} \to \mathbb{R}^2$ be the linear transformation given in Example 6.1.8 by the formula $T(p(x)) = (p(2), p'(4))$. Find $\mathrm{Ker}(T)$, $\mathrm{Rng}(T)$, and their dimensions.

$p'(x) = 2ax + b$

$$T : P_2(\mathbb{R}) \longrightarrow \mathbb{R}^2$$

$$T(p(x)) = (p(2), p'(4))$$

$$\mathrm{Ker}(T) = \{ p \in P_2(\mathbb{R}) : T(p) = 0 \}$$

$$= \{ p(x) \in P_2(\mathbb{R}) : p(2) = 0, \ p'(4) = 0 \}$$

$$= \left\{ ax^2 + bx + c : \begin{array}{l} 4a + 2b + c = 0 \\ 8a + b = 0 \end{array} \right\}$$

$$= \{ ax^2 - 8ax + 12a : a \in \mathbb{R} \}$$

$$= \operatorname{Span}\left(x^2 - 8x + 12 \right)$$

$$\Rightarrow \dim \operatorname{Ker}(T) = 1$$

$$\dim\left(\operatorname{Rng}(T)\right) = \dim V - \dim\left(\operatorname{Ker}(T)\right)$$

$$= 3 - 1 = 2$$

GEN. [RANK -NULLITY]

$$\Rightarrow \operatorname{Rng}(T) = \mathbb{R}^2 \quad \left(\because \operatorname{Rng}(T) \subseteq \mathbb{R}^2, \ \dim \operatorname{Rng}(T) = \dim \mathbb{R}^2 = 2 \right)$$

BREAK

TILL

10:05 AM

7.1 EIGENVALUE / EIGENVECTOR

MATRIX MULT.

$A : \mathbb{R}^n \to \mathbb{R}^n$

DEFINITION 7.1.1

Let A be an $n \times n$ matrix. Any values of λ for which

$v \neq 0$

$$Av = \lambda v \tag{7.1.1}$$

has *nontrivial* solutions **v** are called **eigenvalues** of A. The corresponding *nonzero* vectors **v** are called **eigenvectors** of A.

SCALAR MULTIPLI-CATION.

$$A\vec{v} \overset{?}{=} \lambda \vec{v}$$

Example 7.1.2 Let $A = \begin{bmatrix} -2 & 5 \\ 6 & -1 \end{bmatrix}$. Show that $\mathbf{v}_1 = (-1, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = -7$, and show that $\mathbf{v}_2 = (5, 6)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 4$.

$$A = \begin{bmatrix} -2 & 5 \\ 6 & -1 \end{bmatrix} \qquad \vec{v}_1 = (-1, 1)$$

$$A\vec{v}_1 = \begin{bmatrix} -2 & 5 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} (-2)(-1) + (5)(1) \\ (6)(-1) + (-1)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -7 \end{bmatrix}$$

$$A\vec{v}_1 = (7, -7) = -7(-1, 1) = -7\vec{v}_1$$

$$A\vec{v_2} = \begin{bmatrix} -2 & 5 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} (-2)(5) + (5)(6) \\ (6)(5) + (-1)(6) \end{bmatrix}$$

$$= \begin{bmatrix} 20 \\ 24 \end{bmatrix}$$

$$A\vec{v_2} = (20, 24) = 4(5, 6) = 4\vec{v_2}$$

Q - HOW TO FIND EIGENVECTORS / EIGENVALUES?

WHEN DOES $A\vec{v} = \lambda\vec{v}$ HAVE NON-TRIVIAL SOLUTIONS IN $\vec{v}$?

$$\vec{v} = I\vec{v} \qquad (I \to n \times n \text{ IDENTITY})$$

$$A\vec{v} = \lambda\vec{v} \Rightarrow A\vec{v} - \lambda\vec{v} = 0$$

$$\Rightarrow A\vec{v} - \lambda(I\vec{v}) = 0$$

$$\Rightarrow (A - \lambda I)\vec{v} = 0 \Leftrightarrow v \in \text{NULLSPACE } (A - \lambda I)$$

$B\vec{v} = 0$ $\qquad$ B — SQUARE

NON-TRIVIAL SOLN. $\Leftrightarrow$ $\det(B) = 0$

$\Updownarrow$

$\text{Rank } B \neq n$

$B = (A - \lambda I)$

$\det(A - \lambda I) = 0$

Solution to the Eigenvalue/Eigenvector Problem

1. Find all scalars λ with $\det(A - \lambda I) = 0$. These are the eigenvalues of A.
2. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the *distinct* eigenvalues obtained in (1), then solve the k systems of linear equations

$$(A - \lambda_i I)\mathbf{v}_i = 0, \qquad i = 1, 2, \ldots, k$$

to find all eigenvectors $\mathbf{v}_i$ corresponding to each eigenvalue.

DEFINITION 7.1.3

For a given $n \times n$ matrix A, the polynomial $p(\lambda)$ defined by

$$p(\lambda) = \det(A - \lambda I)$$

POLYNOMIAL IN λ.

is called the **characteristic polynomial** of A, and the equation

$$p(\lambda) = 0$$

is called the **characteristic equation** of A.

Proposition 7.1.4 An $n \times n$ matrix A is invertible if and only if 0 is not an eigenvalue of A.

Pf. A IS INVERTIBLE $\Leftrightarrow \det A \neq 0 \Leftrightarrow \det(A - 0I) \neq 0 \Leftrightarrow 0$ IS NOT AN EIGENVALUE.

Example 7.1.5 Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$.

$$p(\lambda) = \det(A - \lambda I)$$

$$A - \lambda I = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & -1 \\ -5 & -\lambda-1 \end{bmatrix}$$

$$\det(A - \lambda I) = (3-\lambda)(-\lambda-1) - (-1)(-5)$$

$$= (\lambda-3)(\lambda+1) - 5$$

$$= \lambda^2 - 2\lambda - 8$$

$$p(\lambda) = \lambda^2 - 2\lambda - 8 = 0$$

$$= \lambda^2 - 4\lambda + 2\lambda - 8$$

$$= \lambda(\lambda - 4) + 2(\lambda - 4)$$

$$= (\lambda + 2)(\lambda - 4) = 0$$

$$\Rightarrow \lambda = -2, \quad \lambda = 4$$

EIGENVALUES.

$$(A - \lambda I)\vec{v} = 0$$

$$A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}, \quad \lambda = 4$$

$$A - \lambda I = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -5 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -5 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$v_1 + v_2 = 0$$

$v_1 = t$, $v_2 = -t$

$\therefore \vec{v} = (t, -t)$, $t \in \mathbb{R}$

$= t(1, -1)$

$\text{EIGENVECTORS}(A, 4) = \{ t(1,-1) : t \neq 0, t \in \mathbb{R} \}$

$$\lambda = -2$$

$$A - \lambda I = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -5 & 1 \end{bmatrix}$$

$$(A - \lambda I)\vec{v} = 0 \iff \begin{bmatrix} 5 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$5v_1 - v_2 = 0$$

$$v_1 = t, \quad v_2 = 5v_1 = 5t$$

$$\vec{v} = (t, 5t) = t(1, 5)$$

$$\text{EIGENVECTORS}(A, -2) = \{ t(1,5) : \begin{matrix} t \neq 0 \\ t \in \mathbb{R} \end{matrix} \}$$

Example 7.1.6 Find all eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} 5 - \lambda & 12 & -6 \\ -3 & -10 - \lambda & 6 \\ -3 & -12 & 8 - \lambda \end{bmatrix}$$

$$= (5 - \lambda) \begin{vmatrix} -10 - \lambda & 6 \\ -12 & 8 - \lambda \end{vmatrix} - 12 \begin{vmatrix} -3 & 6 \\ -3 & 8 - \lambda \end{vmatrix}$$

$$+ -6 \begin{vmatrix} -3 & -10 - \lambda \\ -3 & -12 \end{vmatrix}$$

$$(5-\lambda)\begin{vmatrix} -10-\lambda & 6 \\ -12 & 8-\lambda \end{vmatrix} - 12\begin{vmatrix} -3 & 6 \\ -3 & 8-\lambda \end{vmatrix}$$

$$+(-6)\begin{vmatrix} -3 & -10-\lambda \\ -3 & -12 \end{vmatrix}$$

$$= (5-\lambda)\left[(-10-\lambda)(8-\lambda) - 6(-12)\right]$$
$$-12\left[-3(8-\lambda) - 6(-3)\right] - 6\left[(-3)(-12) - (-10-\lambda)(-3)\right]$$

$$= (5-\lambda)\left[\lambda^2 + 2\lambda - 8\right] - 12(3\lambda - 6)$$
$$-6\left[-3\lambda + 6\right]$$

$$(5-\lambda)\left[\lambda^2+2\lambda-8\right]-12(3\lambda-6)$$
$$-6\left[-3\lambda+6\right]$$

$$(5-\lambda)(\lambda-2)(\lambda+4)-36(\lambda-2)+18(\lambda-2)$$

$$p(\lambda)=(\lambda-2)\left[(5-\lambda)(\lambda+4)-36+18\right]$$
$$=(\lambda-2)\left[-\lambda^2+\lambda+2\right]$$
$$=-(\lambda-2)^2(\lambda+1) \longrightarrow \begin{array}{l}\lambda=2\\ \lambda=-1\end{array}$$

EIGEN VECTORS OF $\lambda = 2$

$$A - \lambda I = \begin{bmatrix} 5 - \lambda & 12 & -6 \\ -3 & -10 - \lambda & 6 \\ -3 & -12 & 8 - \lambda \end{bmatrix}$$

$$\underset{(\lambda = 2)}{=} \begin{bmatrix} 3 & 12 & -6 \\ -3 & -12 & 6 \\ -3 & -12 & 6 \end{bmatrix}$$

$$(A - \lambda I)\vec{v} = 0 \longrightarrow \begin{bmatrix} 1 & 4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$v_2 = t \quad , \quad v_3 = s$$

$$v_1 = -4v_2 + 2v_3$$

$$= -4t + 2s$$

$$\vec{v} = (-4t + 2s, t, s)$$

EIGENVECTORS (OF $\lambda = 2$) $= \left\{ \underbrace{(-4t+2s, t, s)}_{\text{2 PARAMS.}} : t, s \in \mathbb{R} \text{ EITHER } t \neq 0 \text{ OR } s \neq 0 \right\}$

EIGEN VECTORS OF $\lambda = -1$

$$A - \lambda I = \begin{bmatrix} 5 - \lambda & 12 & -6 \\ -3 & -10 - \lambda & 6 \\ -3 & -12 & 8 - \lambda \end{bmatrix}$$

$$\underset{(\lambda = -1)}{=} \begin{bmatrix} 6 & 12 & -6 \\ -3 & -9 & 6 \\ -3 & -12 & 9 \end{bmatrix}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\begin{bmatrix} 6 & 12 & -6 \\ -3 & -9 & 6 \\ -3 & -12 & 9 \end{bmatrix}$$

$$\Big\downarrow M_2(1/6)$$

$$\begin{bmatrix} 1 & 2 & -1 \\ -3 & -9 & 6 \\ -3 & -12 & 9 \end{bmatrix} \underset{A_{13}(3)}{\overset{A_{12}(3)}{\sim\!\!\!\longrightarrow}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{bmatrix}$$

$$\Big\downarrow M_2(-3) \; ; \; A_{23}(6)$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{aligned} v_1 + 2v_2 - v_3 &= 0 \\ v_2 - v_3 &= 0 \end{aligned}$$

$v_3 = t \Rightarrow v_2 = t, \ v_1 = -t$

$$\text{EIGENVECTOR (OF } -1) = \{ (\underbrace{-t, t, t}_{\text{1 PARAM.}}) : t \in \mathbb{R}, \ t \neq 0 \}$$

Example 7.1.7 Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

DEFECTIVE.

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)^2 - 1 \times 0 = (1-\lambda)^2$$

EIGEN VALUE. $\lambda = 1$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$

$$\underset{(\lambda=1)}{=} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(A - \lambda I)\vec{v} = 0 \iff \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff v_2 = 0$$

$$\{\vec{v} = (t, 0) : t \in \mathbb{R},\ t \neq 0\} \longrightarrow 1 \text{ PARAMETER FAMILY}$$

Example 7.1.10 Let λ be an eigenvalue of the matrix A with corresponding eigenvector $\mathbf{v}$. Prove that λ^2 is an eigenvalue of A^2 with corresponding eigenvector $\mathbf{v}$.

Pf. SUPPOSE $(\lambda, \vec{v})$ IS AN EIGENPAIR (FOR A)

$$A\vec{v} = \lambda\vec{v} \quad - \text{I}$$

OTOH:

$$A^2\vec{v} = (A \cdot A)\vec{v}$$
$$= A[A\vec{v}]$$
$$= A[\lambda\vec{v}] \quad (\text{BY I})$$
$$= \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) \quad (\text{BY I})$$
$$= \lambda^2\vec{v}$$

$$A^2\vec{v} = \lambda^2\vec{v}$$
$$\Updownarrow$$

$(\lambda^2, \vec{v})$ IS AN EIGENPAIR FOR A^2.

Example 7.1.11 Let λ and $\mathbf{v}$ be an eigenvalue/eigenvector pair for the $n \times n$ matrix A. If k is an arbitrary real number, prove that $\mathbf{v}$ is also an eigenvector of the matrix $A - kI$ corresponding to the eigenvalue $\lambda - k$.

LET $(\lambda, \vec{v})$ BE AN EIGENPAIR FOR A.

$$A\vec{v} = \lambda\vec{v} \quad \text{①}$$

$$\begin{aligned}(A - kI)\vec{v} &= A\vec{v} - (kI)\vec{v} \\ &= A\vec{v} - k(I\vec{v}) \\ &= A\vec{v} - k\vec{v} \qquad \text{(BY ①)} \\ &= \lambda\vec{v} - k\vec{v} = (\lambda - k)\vec{v}\end{aligned}$$

$(A - kI)\vec{v} = (\lambda - k)\vec{v}$

$\Updownarrow$

$(\lambda - k, \vec{v})$ IS AN EIGENPAIR FOR $A - kI$.

(MY DEFN OF DEFECTIVE)

A MATRIX IS DEFECTIVE IF IT HAS AN EIGENVALUE λ_j W/ MULTIPLICITY m_j IN

$$p(\lambda) = \det(A - \lambda I) = 0$$

s.t.

dim (EIGEN-SPACE)

OF INDEPENDENT PARAM. IN THE EIGENVALUES OF λ_j $< m_j$

§ 7.2 GENERAL RESULTS FOR EIGENVALUES & EIGENVECTORS

NEXT TIME:

(BOOK'S DEFN OF DEFECTIVE)

DEFINITION 7.2.7

An $n \times n$ matrix A that has n linearly independent eigenvectors is called **nondefective**. In such a case, we say that A has a **complete set of eigenvectors**. If A has less than n linearly independent eigenvectors, it is called **defective**.

NEXT TIME : BOTH DEFINITIONS ARE EQUIVALENT.

Theorem 7.2.11 An $n \times n$ matrix A is nondefective if and only if the dimension of each eigenspace is the same as the algebraic multiplicity m_i of the corresponding eigenvalue; that is, if and only if $\dim[E_i] = m_i$ for each i.