MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
T – 9:00 PM – 10:00 PM (ET)
F – 3:00 PM – 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1-17 ARE UPLOADED.
2. WW 08, 09 - IS DUE WED (27th JULY) AT 11:00 PM ET
   WW 10, 11 - IS DUE MON (1st AUG) AT 11:00 PM ET
3. HARD WEBWORK DEADLINE : FRIDAY, 5th AUG
4. MIDTERM 2 IS GRADED (REGRADE REQUESTS BY SAT)
5. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE!

$A \in M_n(\mathbb{R})$

(NON-TRIVIAL)

$\lambda \in \mathbb{R}$ s.t. $A\vec{v} = \lambda \vec{v}$ HAS A SOLUTION $\vec{v} \in \mathbb{R}^n$

$(A - \lambda I)\vec{v} = 0$

Solution to the Eigenvalue/Eigenvector Problem

1. Find all scalars λ with $\det(A - \lambda I) = 0$. These are the eigenvalues of A.
2. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the *distinct* eigenvalues obtained in (1), then solve the k systems of linear equations

$$(A - \lambda_i I)\mathbf{v}_i = 0, \qquad i = 1, 2, \ldots, k$$

to find all eigenvectors $\mathbf{v}_i$ corresponding to each eigenvalue.

§ 7.2 GENERAL RESULTS FOR EIGENVALUES & EIGENVECTORS

DEFINITION 7.2.1

Let A be an $n \times n$ matrix. For a given eigenvalue λ_i, let E_i denote the set of *all* vectors $\mathbf{v}$ satisfying $A\mathbf{v} = \lambda_i \mathbf{v}$. Then E_i is called the $\boxed{\textbf{eigenspace}}$ of A corresponding to the eigenvalue λ_i. Thus, E_i is the solution set to the linear system $(A - \lambda_i I)\mathbf{v} = \mathbf{0}$.

$\vec{v}, \vec{w}$ ARE EITHER $\vec{0}$ OR AN EIGENVECTOR (OF λ)

$\vec{v}+\vec{w}$, $c\vec{v}$ ARE ALSO.

$$A(c\vec{v}) = cA\vec{v} = c(\lambda\vec{v}) = \lambda(c\vec{v})$$

$$A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w} = \lambda\vec{v} + \lambda\vec{w} = \lambda(\vec{v}+\vec{w})$$

Example 7.2.2 Determine all eigenspaces for the matrix $A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$.

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -1 \\ -5 & -1-\lambda \end{bmatrix}$$

$$p(\lambda) = (\lambda-4)(\lambda+2)$$

$\Rightarrow$ EIGENVALUES $= 4, -2$

$$= (3-\lambda)(-1-\lambda) - (-1)(-5)$$

$$= \lambda^2 - 2\lambda - 3 - 5$$

$$= \lambda^2 - 2\lambda - 8$$

$$= (\lambda^2 - 4\lambda) + (2\lambda - 8)$$

$$= (\lambda-4)(\lambda+2)$$

$$\text{EIGENSPACE}(-2) = (A-(-2)I)\vec{v} = 0$$

$$\begin{bmatrix} 3-(-2) & -1 \\ -5 & -1-(-2) \end{bmatrix}\vec{v} = 0$$

$$\Rightarrow \begin{bmatrix} 5 & -1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$5v_1 - v_2 = 0$$

$$v_1 = t$$

$$v_2 = 5v_1 = 5t$$

$$\text{EIGENSPACE} = \{(t, 5t) : t \in \mathbb{R}\}$$

$$(t, 5t) = t(1,5)$$

EIGENSPACE $= \text{Span}\{(1,5)\}$

EIGENSPACE OF $\lambda = 4$

$$\begin{bmatrix} 3-4 & -1 \\ -5 & -1-4 \end{bmatrix} \vec{v} = 0$$

$$\begin{bmatrix} -1 & -1 \\ -5 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} =$$

$-v_1 - v_2 = 0$ $\quad v_1 = t \Rightarrow v_2 = -v_1 = -t$

$(t, -t) = t(1, -1)$

EIGENSPACE OF $\lambda = 4$ $= \{ (t, -t) : t \in \mathbb{R} \}$

$= \text{Span} \{ (1, -1) \}$

Example 7.2.4 Determine all eigenspaces and their dimensions for the matrix

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix}.$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -1 & 0 \\ 0 & 2-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{bmatrix}$$

$$= (-1)^{3+3} (2-\lambda) \begin{vmatrix} 3-\lambda & -1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 (3-\lambda)$$

$$p(\lambda) = (2-\lambda)^2(3-\lambda)$$

EIGENVALUES $= \{2, 3\}$

$$\lambda = 3$$

$$(A - 3I)\vec{v} = 0 \Rightarrow \begin{bmatrix} 3-3 & -1 & 0 \\ 0 & 2-3 & 0 \\ -1 & 1 & 2-3 \end{bmatrix} \vec{v} = 0$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$\xrightarrow{\text{EROs}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v_1 - v_2 + v_3 = 0$$

$$v_2 = 0$$

$$v_3 = t \Rightarrow v_1 = v_2 - v_3 = 0 - t = -t$$

EIGENSPACE OF $\lambda = 3$ $= \{(-t, 0, t) : t \in \mathbb{R}\} = \underbrace{\text{Span}\{(-1, 0, 1)\}}_{\dim = 1}$

$\lambda = 2$

$(A - 2I)\vec{v} = 0 \Rightarrow \begin{bmatrix} 3-2 & -1 & 0 \\ 0 & 2-2 & 0 \\ -1 & 1 & 2-2 \end{bmatrix} \vec{v} = 0$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} \vec{v} = 0$

$\{v_1, v_2, v_3\}$

$\downarrow A_{13}(1)$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = 0 \longrightarrow$

$v_1 - v_2 = 0$

$v_2 = t, \quad v_3 = s$

$v_1 = t$

$v_1 = v_2 = t \ , \ v_3 = s$

EIGENSPACE OF $\lambda = 2$ $= \{(t, t, s) : t, s \in \mathbb{R}\}$

$$(t,t,s) = (t,t,0) + (0,0,s)$$

$$= t(1,1,0) + s(0,0,1)$$

$$= \text{Span}\{(1,1,0), (0,0,1)\}$$

$\dim = 2$

ALGEBRAIC MULTIPLICITY

Theorem 7.2.3 Let λ_i be an eigenvalue of A of multiplicity $\boxed{m_i}$ and let E_i denote the corresponding eigenspace. Then

GEOMETRIC MULTIPLICITY

1. For each i, E_i is a subspace of $\mathbb{C}^n$.
2. If $\boxed{n_i}$ denotes the dimension of E_i, then $\boxed{1 \leq n_i \leq m_i}$ for each i. In words, the dimension of the eigenspace corresponding to λ_i is at most the multiplicity of λ_i.

$$p(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

[FUNDAMENTAL THEOREM OF ALGEBRA].

Theorem 7.2.5 Eigenvectors corresponding to *distinct* eigenvalues are linearly independent.

$\vec{v}_1, \dots, \vec{v}_k \qquad \lambda_1, \dots, \lambda_k$

$\mathrm{I} \rightarrow \boxed{c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = 0} \qquad A\vec{v}_j = \lambda \vec{v}_j \ (1 \le j \le k)$

$A(c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k) = 0$

$\Rightarrow \quad c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + \cdots + c_k A\vec{v}_k = 0$

$= c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + \cdots + c_k \lambda_k \vec{v}_k = 0$

$\lambda_k \neq 0 \Rightarrow \quad \frac{c_1 \lambda_1}{\lambda_k} \vec{v}_1 + \frac{c_2 \lambda_2}{\lambda_k} \vec{v}_2 + \cdots + \frac{c_{k-1} \vec{v}_{k-1}}{\lambda_{k-1}} + c_k v_k = 0$

II

$$\left(\cancel{c_1 - \frac{c_1\lambda_1}{\lambda_k}}^{\,0}\right)\vec{v}_1 + \left(\cancel{c_2 - \frac{c_2\lambda_2}{\lambda_k}}^{\,0}\right)\vec{v}_2 + \cdots + \left(c_{k-1} \cancel{-} \frac{c_{k-1}\lambda_{k-1}}{\lambda_k}\right)^{0}$$

$$\vec{v}_{k-1} = 0$$

$$\Rightarrow \{\vec{v}_1, \dots, \vec{v}_{k-1}\}$$

IS L.D.

$$\Rightarrow \{\vec{v}_1, \dots, \vec{v}_{k-2}\}$$

IS L.D.

$$\vdots$$

$\{\vec{v}_1\}$ IS L.D.

Corollary 7.2.6 Let $E_1, E_2, \ldots, E_k$ denote the eigenspaces of an $n \times n$ matrix A. In each eigenspace, choose a set of linearly independent eigenvectors, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ denote the union of the linearly independent sets. Then $\underbrace{\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}}$ is linearly independent.

DEFINITION 7.2.7

An $n \times n$ matrix A that has n linearly independent eigenvectors is called **nondefective**. In such a case, we say that A has a **complete set of eigenvectors**. If A has less than n linearly independent eigenvectors, it is called **defective**.

Corollary 7.2.10 If an $n \times n$ matrix A has n *distinct* eigenvalues, then it is nondefective.

PF: $\lambda_1, \ldots, \lambda_n$ ARE DISTINCT

$\vec{v}_1, \ldots, \vec{v}_n$ ARE L.I.

$\Rightarrow$ A IS NON-DEFECTIVE.

Theorem 7.2.11 An $n \times n$ matrix A is nondefective if and only if the dimension of each eigenspace is the same as the algebraic multiplicity m_i of the corresponding eigenvalue; that is, if and only if $\dim[E_i] = m_i$ for each i.

UPSHOT: (1) GEOM. MULT. = ALG. MULT. FOR EVERY λ

$\Rightarrow$ MATRIX IS NONDEFECTIVE.

(2) GEOM. MULT. < ALG. MULT. FOR SOME λ

$\Rightarrow$ MATRIX IS DEFECTIVE.

BREAK TILL

10:00 AM

§ 4.5 LINEAR INDEPENDENCE

$$V = C^0(I) \supseteq C^1(I) \supseteq C^2(I) \supseteq \dots$$

$$c_1 f_1 + c_2 f_2 + \dots + c_k f_k = \underbrace{0} \Rightarrow c_1 = c_2 = \dots = c_k$$

$\rightarrow$ THE FUNCTION.

DEFINITION 4.5.19

The set of functions $\{f_1, f_2, \dots, f_k\}$ is **linearly independent on an interval** I if and only if the only values of the scalars $c_1, c_2, \dots, c_k$ such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) = 0, \quad \text{for all } x \in I, \tag{4.5.5}$$

are $c_1 = c_2 = \cdots = c_k = 0$.

f, g

$f = g$

$f(x) = g(x) \quad \forall x$

k columns

DEFINITION 4.5.20

Let $f_1, f_2, \dots, f_k$ be functions in $C^{k-1}(I)$. The **Wronskian** of these functions is the order k determinant defined by

$$W[f_1, f_2, \dots, f_k](x) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_k(x) \\ f_1'(x) & f_2'(x) & \dots & f_k'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \dots & f_k^{(k-1)}(x) \end{vmatrix}.$$

k rows

WRONSKIAN OF $(f_1, \dots, f_k)$

$$W[\sin x, \cos x] = -W[\cos x, \sin x]$$

Example 4.5.21 If $f_1(x) = \sin x$ and $f_2(x) = \cos x$ on $(-\infty, \infty)$, then

$k = 2$

$k - 1 = 1$

$f_1'(x) = \cos x$

$f_2'(x) = -\sin x$

$$W[f_1, f_2](x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= (\sin x)(-\sin x) - (\cos x)(\cos x)$$

$$= -\sin^2 x - \cos^2 x$$

$$= -1$$

Example 4.5.22 If $f_1(x) = x$, $f_2(x) = \dfrac{1}{x^2}$, and $f_3(x) = \dfrac{1}{x^4}$ on $(0, \infty)$,

$k = 3$

$k - 1 = 2$

$f_1'(x) = 1$ $\quad$ $f_2'(x) = -2x^{-3}$ $\quad$ $f_3'(x) = -4x^{-5}$

$f_1''(x) = 0$ $\quad$ $f_2''(x) = 6x^{-4}$ $\quad$ $f_3''(x) = 20x^{-6}$

$$W[f_1, f_2, f_3](x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix} = \begin{vmatrix} x & x^{-2} & x^{-4} \\ 1 & -2x^{-3} & -4x^{-5} \\ 0 & 6x^{-4} & 20x^{-6} \end{vmatrix}$$

$$\begin{vmatrix} x & x^{-2} & x^{-4} \\ 1 & -2x^{-3} & -4x^{-5} \\ 0 & 6x^{-4} & 20x^{-6} \end{vmatrix} = x \cdot \begin{vmatrix} -2x^{-3} & -4x^{-5} \\ 6x^{-4} & 20x^{-6} \end{vmatrix} - 1 \cdot \begin{vmatrix} x^{-2} & x^{-4} \\ 6x^{-4} & 20x^{-6} \end{vmatrix}$$

$$= x\left(-40x^{-9} - (-24x^{-9})\right)$$

$$- \left(20x^{-8} - 6x^{-8}\right)$$

$$= -16x^{-8} - 14x^{-8} = -30x^{-8}$$

Theorem 4.5.23 Let $f_1, f_2, \ldots, f_k$ be functions in $C^{k-1}(I)$. If $W[f_1, f_2, \ldots, f_k]$ is nonzero at some point x_0 in I, then $\{f_1, f_2, \ldots, f_k\}$ is linearly independent on I.

$$c_1 f_1 + \cdots + c_k f_k = 0$$

DIFF.

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) = 0$$

$$c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_k f_k'(x) = 0$$

$$c_1 f^{(2)}(x) + \cdots + c_k f_k^{(2)}(x) = 0$$

$$\vdots$$

$$c_1 f_1^{(k-1)}(x) + c_2 f^{(k-1)}(x) + \cdots + c_k f_k^{(k-1)}(x) = 0$$

$$\begin{bmatrix} f_1(x) & \cdots & f_k(x) \\ f_1'(x) & \cdots & f_k'(x) \\ \vdots & & \vdots \\ f_1^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A\vec{c} = \vec{0}$$

$A \to$ NUMBERS DEPENDING ON x.

IF $\det A \neq 0$ FOR ANY x,

$$\vec{c} = \vec{0}$$

$\therefore \quad W[f_1, \ldots, f_k](x_0) = 0 \quad$ FOR ANY $x_0 \in I$

$\Rightarrow \quad c_1 = c_2 = \cdots = c_k = 0$

$\therefore \quad f_1, \ldots, f_k$ IS L.I.

$$\boxed{\text{If } W[f_1, f_2, \ldots, f_k](x) = 0 \text{ for all } x \text{ in } I\text{, Theorem 4.5.23 gives no information as to the linear dependence or independence of } \{f_1, f_2, \ldots, f_k\} \text{ on } I.}$$

Example 4.5.24 Determine whether the following functions are linearly dependent or linearly independent on $I = (-\infty, \infty)$.

(a) $f_1(x) = e^x$, $f_2(x) = x^2 e^x$.

(b) $f_1(x) = x$, $f_2(x) = x + x^2$, $f_3(x) = 2x - x^2$.

(c) $f_1(x) = x^2$, $f_2(x) = \begin{cases} 2x^2, & \text{if } x \geq 0, \\ -x^2, & \text{if } x < 0. \end{cases}$ $\rightarrow$ $W[f_1, f_2](x) = 0$ $\forall x$

BUT f_1 & f_2 ARE L.I.

$f_3 = 3f_1 - f_2$

$2x - x^2 = f_3$

$3f_1 - f_2 = (3x) - (x + x^2)$

$= 2x - x^2$

$k = 3$

$k-1 = 2$

$f_1(x) = x$ $\quad$ $f_2(x) = x + x^2$ $\quad$ $f_3 = 2x - x^2$

$f_1'(x) = 1$ $\quad$ $f_2'(x) = 1 + 2x$ $\quad$ $f_3'(x) = 2 - 2x$

$f_1''(x) = 0$ $\quad$ $f_2''(x) = 2$ $\quad$ $f_3''(x) = -2$

$$W[f_1, f_2, f_3](x) = \begin{vmatrix} x & x + x^2 & 2x - x^2 \\ 1 & 1 + 2x & 2 - 2x \\ 0 & 2 & -2 \end{vmatrix}$$

$$\begin{vmatrix} x & x+x^2 & 2x-x^2 \\ 1 & 1+2x & 2-2x \\ 0 & 2 & -2 \end{vmatrix} \xrightarrow{CA_{23}(1)} \begin{vmatrix} x & x+x^2 & 3x \\ 1 & 1+2x & 3 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= (-1)^{2+3}\, 2 \begin{vmatrix} x & 3x \\ 1 & 3 \end{vmatrix}$$

$$= (-1)^{2+3}\, 2 \,(3x - 3x)$$

$$= 0$$

$$f_1(x) = e^x \qquad f_2(x) = x^2 e^x$$

$$f_1'(x) = e^x \qquad f_2'(x) = 2x e^x + x^2 e^x$$

$$= (2x + x^2) e^x$$

$$W[f_1, f_2](x) = \begin{vmatrix} e^x & x^2 e^x \\ e^x & (2x+x^2) e^x \end{vmatrix}$$

$$A_{12}(-1)$$

$$= \begin{vmatrix} e^x & x^2 e^x \\ 0 & 2x e^x \end{vmatrix} = 2x e^{2x}$$

$$W[f_1, f_2](1) = 2 \cdot 1 \cdot e^{2 \cdot 1} = 2e^2 \neq 0$$

$\Rightarrow$ e^x & $x^2 e^x$ ARE L.I.

§8.1 GENERAL THEORY FOR LINEAR DIFFERENTIAL EQUATIONS

$$D : C^1(I) \longrightarrow C^0(I), \quad Df = f'$$

LINEAR

$$D(f+g) = (f+g)' = f' + g' = Df + Dg$$

$$D(cf) = (cf)' = cf' = cD(f)$$

$$D^k : C^k(I) \longrightarrow C^0(I)$$

$$D^k f = \frac{d^k}{dx^k} f$$

$$D^k = D \circ D^{k-1}$$

$f \in C^k(I)$

$Df, D^2f, \ldots, D^kf$

LINEAR DIFFERENTIAL OPERATOR:

$$L = D^n + \underbrace{a_1}_{a_1(x)} D^{n-1} + \cdots + \underbrace{a_{n-1}}_{a_{n-1}(x)} D + \underbrace{a_n}_{a_n(x)}$$

$$L(f) = \left(D^n + a_1 D^{n-1} + \cdots + a_n\right) f$$

$$= D^n f + a_1 D^{n-1} f + a_2 D^{n-2} f + \cdots + a_n f$$

$$L : C^n(I) \longrightarrow C^0(I)$$

CRUCIAL:

1. D IS LINEAR
2. L IS LINEAR.

Example 8.1.1 If $L = D^2 + 4xD - 3x$, then

$$Ly = y'' + 4xy' - 3xy,$$

$$Ly = (D^2 + 4xD - 3x)\, y$$
$$= D^2 y + 4xDy - 3xy$$
$$= y'' + 4xy' - 3xy$$

$$L(\sin x) = (D^2 + 4xD - 3x)\sin x$$
$$= D^2(\sin x) + 4x\, D(\sin x) - 3x\sin x$$

$$L(\sin x) = -\sin x + 4x\cos x - 3x\sin x$$

$$L(x^2) = (D^2 + 4xD - 3x)\, x^2 = D^2(x^2) + 4x\, D(x^2) - (3x)x^2$$
$$= 2 + (4x)(2x) - 3x^3$$

$$\boxed{L(x^2) = -3x^3 + 8x^2 + 2}$$

$$L : C^1 \to C^0$$

Example 8.1.2 Determine the kernel of the linear differential operator $L = D - 2x$.

$$\ker L = \{ y \in C^1 : Ly = 0 \}$$

$$Ly = 0 \Rightarrow (D - 2x)\, y = 0$$

$$(D - 2x)\, y = Dy - 2xy = y' - 2xy$$

$$Ly = 0 \Leftrightarrow \boxed{y' - 2xy = 0} \to \text{SOLVE}$$

$$\frac{dy}{dx} = 2xy \Leftrightarrow \frac{dy}{y} = 2x\,dx$$

$$\int \frac{dy}{y} = \int 2x\,dx$$

$$\ln y = x^2 + C$$

$$\Rightarrow \quad y = e^{x^2 + C} = e^C \cdot e^{x^2}$$

k

$$y = k \cdot e^{x^2}$$

$$\ker L = \{ k e^{x^2} : k \in \mathbb{R} \} = \operatorname{span}(e^{x^2})$$

UNIQUENESS & EXISTENCE

Theorem 8.1.3 Let $a_1, a_2, \ldots, a_n$, and F be functions that are continuous on an interval I. Then, for any x_0 in I, the initial-value problem

$$Ly = F(x)$$
$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

has a unique solution on I.

$Ly = F(x)$ $\qquad$ $L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$

$$Ly = F(x) \iff \boxed{y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = F(x)}$$

(nth ORDER LINEAR ODE)

HOMOGENEOUS LINEAR ODEs

$$y^{(n)} + a_1(x)\,y^{(n-1)} + \dots + a_n(x)\,y = 0$$

$$Ly = 0 \qquad \left(L = D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n \right)$$

$$\Leftrightarrow \quad y \in \ker L$$

SOLUTION SET $= \ker L \longrightarrow$ A VECTOR SUBSPACE OF $C^n(I)$

Theorem 8.1.4 The set of all solutions to the regular nth-order homogeneous linear differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \quad - \text{(I)}$$

on an interval I is a vector space of dimension n.

$n=3$, $I=\mathbb{R}$

$y_1, \dots, y_3$

y_1 IS A SOLN. OF (I) WITH INITIAL VALUE DATA

$y_1(0)=1 \qquad y_1'(0)=0 \qquad y_1''(0)=0$

Y_1 IS A SOLN. OF I WITH INITIAL VALUE DATA

$$y_1(0) = 1 \qquad y_1'(0) = 0 \qquad y_1''(0) = 0$$

Y_2 IS A SOLN. OF I WITH INITIAL VALUE DATA

$$y_2(0) = 0 \qquad y_2'(0) = 1 \qquad y_2''(0) = 0$$

Y_1 IS A SOLN. OF I WITH INITIAL VALUE DATA

$$y_3(0) = 0 \qquad y_3'(0) = 0 \qquad y_3''(0) = 1$$

CLAIM I: $\{y_1, y_2, y_3\}$ IS L.I.

Pf.

$$W[y_1, y_2, y_3](0) = \begin{vmatrix} y_1(0) & y_2(0) & y_3(0) \\ y_1'(0) & y_2'(0) & y_3'(0) \\ y_1''(0) & y_2''(0) & y_3''(0) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\Rightarrow$ y_1, y_2, y_3 IS L.I.

<u>CLAIM 2</u>: $\{y_1, y_2, y_3\}$ IS SPANNING.

$y \in$ SOLN. SPACE.

$$y(0) = c_1$$
$$y'(0) = c_2$$
$$y''(0) = c_3$$

$$z = c_1 y_1 + c_2 y_2 + c_3 y_3$$

$$z(0) = c_1 \overset{1}{y_1(0)} + \cancel{c_2 y_2(0)} + \cancel{y_3(0)} = c_1$$

$$z'(0) = \cancel{c_1 y_1'(0)} + c_2 \underbrace{y_2'(0)}_{1} + c_3 \cancel{y_3'(0)} = c_2$$

$$z''(0) = c_1 \cancel{y_1''}(0) + c_2 \cancel{y_2''(0)} + c_3 \underbrace{y_3''(0)}_{1} = c_3$$

$$z(0) = c_1 = y(0)$$

$$z'(0) = c_2 = y'(0)$$

$$z''(0) = c_3 = y''(0)$$

$$\Rightarrow \quad z = y$$

$y_1(0) = 1$, $y_1'(0) = 0$, $y_1''(0) = 0$

$y_2(0) = 0$, $y_2'(0) = 1$, $y_2''(0) = 0$

$y_3(0) = 0$, $y_3'(0) = 0$, $y_3''(0) = 1$

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 \in \mathrm{Span}(y_1, \ldots, y_3)$$

$$\text{SOL'N. SPACE} = \mathrm{Span}(y_1, \ldots, y_3)$$

$$\Rightarrow \text{DIM} = 3.$$

$\therefore$ GENERAL SOLN.

$$y(x) = c_1 y_1(x) + \dots + c_n y_n(x) \qquad c_j \in \mathbb{R}$$

WHERE $y_j \longrightarrow$ L.I. SOLUTIONS.

Example 8.1.5 Determine all solutions to the differential equation $y'' - 2y' - 15y = 0$ of the form $y(x) = e^{rx}$, where r is a constant. Use your solutions to determine the general solution to the differential equation.

$$L = D^2 - 2D - 15$$

$$y'' - 2y' - 15y = 0$$

$$y(x) = e^{rx} \rightarrow y'(x) = r e^{rx} \rightarrow y''(x) = r^2 e^{rx}$$

$$(r^2 e^{rx}) - 2(r e^{rx}) - 15 e^{rx} = 0$$

$$\Rightarrow (r^2 - 2r - 15) \boxed{e^{rx}} = 0 \Rightarrow r^2 - 2r - 15 = 0$$

$$e^{rx} \neq 0$$

$$r^2 - 2r - 15 = 0$$

$$(r-5)(r+3) = 0$$

$$r = 5, \quad r = -3$$

$$y_1 = e^{5x} \qquad (r=5)$$

$$y_2 = e^{-3x} \qquad (r=-3)$$

$$W[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{5x} & e^{-3x} \\ 5e^{5x} & -3e^{-3x} \end{vmatrix}$$

$$\begin{vmatrix} e^{5x} & e^{-3x} \\ 5e^{5x} & -3e^{-3x} \end{vmatrix} = (e^{5x})(-3e^{-3x}) - (5e^{5x})(e^{-3x})$$

$$= -3e^{2x} - 5e^{2x}$$

$$W[y_1, y_2](x) = -8e^{2x}$$

$$W[y_1, y_2](0) = -8 \neq 0$$

$\Rightarrow y_1, y_2$ IS L.I.

GENERAL SOLN $\Rightarrow$ $y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^{5x} + c_2 e^{-3x}$