MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
T – 9:00 PM – 10:00 PM (ET)
F – 3:00 PM – 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEB PAGE

https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1-18 ARE UPLOADED.

2. WW 08, 09 - WAS DUE WED (27th JULY) AT 11:00 PM ET
   WW 10, 11 - IS DUE MON (1st AUG) AT 11:00 PM ET

3. HARD WEBWORK DEADLINE: FRIDAY, 5th AUG

4. MIDTERM 2 IS GRADED (REGRADE REQUESTS BY SAT)

5. REMINDER: PLEASE KEEP VIDEOS ON, IF POSSIBLE!

HOMOGENEOUS LINEAR ODEs

$f(x) = 0$

Theorem 8.1.4 The set of all solutions to the regular nth-order homogeneous linear differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \rightarrow Ly = 0$$

on an interval I is a vector space of dimension n.

$L = D^n + a_1 D^{n-1} + \cdots + a_n$

$(L : C^n(I) \to C^0(I))$

$c_j \in \mathbb{R}$

$\therefore$ GENERAL SOLN.

$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$

WHERE $y_j \longrightarrow$ L.I. SOLUTIONS.

Theorem 8.1.6 Let $y_1, y_2, \ldots, y_n$ be solutions to the regular nth-order differential equation $Ly = 0$ on an interval I, and let $W[y_1, y_2, \ldots, y_n](x)$ denote their Wronskian. If $W[y_1, y_2, \ldots, y_n](x_0) = 0$ at some point x_0 in I, then $\{y_1, y_2, \ldots, y_n\}$ is *linearly dependent* on I.

$\cos x$, $\cos x + \sin x$, $\sin x$ $\rightarrow$ $y'' + y = 0$

$W[\cos x, \cos x + \sin x, \sin x] = 0 \quad \forall x$

> The vanishing or nonvanishing of the Wronskian on an interval I completely characterizes whether *solutions to* $Ly = 0$ are linearly dependent or linearly independent on I.

Example 8.1.7 Verify that $y_1(x) = \cos 2x$ and $y_2(x) = 3(1 - 2\sin^2 x)$ are solutions to the differential equation $y'' + 4y = 0$ on $(-\infty, \infty)$. Determine whether they are linearly independent on $(-\infty, \infty)$.

$$y_1 = \cos 2x$$

$$y_1' = -2\sin 2x$$

$$y_1'' = -4\cos 2x$$

$$y_2 = 3(1-2\sin^2 x)$$

$$y_2' = -12\sin x\cos x$$

$$y_2'' = -12(\cos^2 x) + 12\sin^2 x$$

$$= 12(\sin^2 x - \cos^2 x)$$

$$= -12(1 - 2\sin^2 x)$$

$$y_1'' + 4y_1 = -4\cos 2x + 4\cos 2x = 0, \quad y_2'' + 4y_2 = 0$$

$$W[y_1, y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & 3(1-2\sin^2 x) \\ -2\sin 2x & -12\sin x \cos x \end{vmatrix}$$

$$= (\cos 2x)(-12 \sin x \cos x) + 6 \sin 2x (1-2\sin^2 x)$$

$$W[y_1, y_2](0) = (1)(-12 \cdot 0 \cdot 1) + 6 \cdot 0 (1 - 2 \cdot 0^2)$$

$$= 0$$

NON-HOMOGENEOUS LINEAR ODEs

$$y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_n(x)\, y = F(x)$$

$$Ly = F$$

$$L = D^n + a(x) D^{n-1} + \dots + a_n(x)$$

$$A\vec{x} = \vec{0}$$

$$A\vec{x} = \vec{b}$$

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

Theorem 8.1.8 Let $\{y_1, y_2, \ldots, y_n\}$ be a linearly independent set of solutions to $\boxed{Ly = 0}$ on an interval I, and let $y = y_p$ be any *particular* solution to $Ly = F$ on I. Then every solution to $Ly = F$ on I is of the form

$\to$ dim $= n$

$$y = \underbrace{c_1 y_1 + c_2 y_2 + \cdots + c_n y_n}_{y_c} + y_p,$$

for appropriate constants $c_1, c_2, \ldots, c_n$.

y SOLVES $Ly = F$

$y_p \to$ PARTICULAR SOLN. OF $Ly = F$

$$L(y - y_p) = Ly - Ly_p = F(x) - F(x) = 0$$

$\Rightarrow y - y_p \in \ker L \Rightarrow y - y_p \in \operatorname{Span}\{y_1, \ldots, y_n\}$

Example 8.1.9 Verify that $y_p(x) = 2e^{6x}$ is a particular solution to the differential equation

$$y'' - 2y' - 15y = 18e^{6x}$$

and determine the general solution.

$$y_p = 2e^{6x}, \quad y_p' = 12e^{6x}, \quad y_p'' = 72e^{6x}$$

$$y_p'' - 2y_p' - 15y_p = 72e^{6x} - 2(12e^{6x}) - 15(2e^{6x})$$

$$= 72e^{6x} - 24e^{6x} - 30e^{6x}$$

$$= 18e^{6x}$$

$\Rightarrow$ y_p IS A SOLUTION.

$$y'' - 2y' - 15y = 0$$

$$y = e^{rx}, \quad y' = r e^{rx}, \quad y'' = r^2 e^{rx}$$

$$r^2 e^{rx} - 2r e^{rx} - 15 e^{rx} = 0$$

$$\Rightarrow (r^2 - 2r - 15)\underbrace{e^{rx}}_{\neq 0} = 0$$

$$r^2 - 2r - 15 = 0 \iff (r-5)(r+3) = 0$$

$$r = 5, -3 \qquad y_1 = e^{-3x}, \quad y_2 = e^{5x}$$

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-3x} & e^{5x} \\ -3e^{-3x} & 5e^{5x} \end{vmatrix}$$

$$= 5e^{2x} + 3e^{2x}$$

$$= 8e^{2x} \neq 0$$

$\Rightarrow \{y_1, y_2\}$ IS AN L.I.

$Ly = 0 \Leftrightarrow y = c_1 y_1 + c_2 y_2$

$\therefore Ly = 18e^{6x} \Leftrightarrow y = c_1 y_1 + c_2 y_2 + y_p$

$$= c_1 e^{-3x} + c_2 e^{5x} + 2e^{6x}$$

$$\boxed{y(x) = c_1 e^{-3x} + c_2 e^{5x} + 2e^{6x}}$$

GENERAL SOLN.

Theorem 8.1.10 If $y = u_p$ and $y = v_p$ are particular solutions to $Ly = f(x)$ and $Ly = g(x)$, respectively, then $y = u_p + v_p$ is a solution to $Ly = f(x) + g(x)$.

$u_p \to$ SOLVES $Ly = f$

$v_p \to$ SOLVES $Ly = g$

$L(u_p + v_p) = Lu_p + Lv_p$

$L(u_p + v_p) = f(x) + g(x) \Rightarrow u_p + v_p$ SOLVES $Ly = f + g$

BREAK
TILL
9:35

§ 8.2 CONSTANT COEFFICIENT HOMOGENEOUS LINEAR D.E.

$$Ly = 0$$

$$L = D^n + a_1 D^{n-1} + \cdots + a_n$$

$a_j \to$ CONSTANT.

$$L = P(D) \longrightarrow P(r)$$ (AUXILIARY POLYNOMIAL).

$D \to \frac{d}{dx}$

Example 8.2.1 Write the differential equation $y'' + 5y' - 7y = 0$ as $P(D)y = 0$ for an appropriate polynomial differential operator $P(D)$. Determine the auxiliary polynomial and the auxiliary equation.

$$y'' + 5y' - 7y = D^2 y + 5Dy - 7y$$

$$= \underbrace{(D^2 + 5D - 7)}_{L = P(D)} y$$

$$P(D) = D^2 + 5D - 7$$

$$P(D)y = (D^2 + 5D - 7) y$$

$$= D^2 y + 5Dy - 7y$$

$$= y'' + 5y' - 7y$$

$$P(r) = r^2 + 5r - 7$$

(AUXILIARY POLYNOMIAL)

$P(\lambda) = 0$ (AUXILIARY EQUATION)

$\lambda^2 + 5\lambda - 7 = 0$

$$\underbrace{D^2 f(x)}_{f''(x)}$$

$$L = f(x)\, D^2$$

$$L : C^2(I) \longrightarrow C(I)$$

$$Ly = f(x)\, D^2 y$$

$$= f(x)\, y''$$

$$L_1 = D \qquad\qquad L_2 = xD$$

$$L_1 y = Dy = y' \qquad\qquad L_2 y = xDy$$

$$= xy'$$

$$L_1 L_2 y = L_1 [L_2 y] = L_1 [xy'] = D(xy')$$

$$L_2 L_1 y = L_2 [L_1 y] \qquad\qquad = \frac{d}{dx}(xy')$$

$$= xD(y') \qquad\qquad = y' + xy''$$

$$= x \frac{d}{dx} y' = xy''$$

Theorem 8.2.2 If $P(D)$ and $Q(D)$ are polynomial differential operators, then

$$P(D)Q(D) = Q(D)P(D).$$

$$P(r) = r^2 + 1$$

$$Q(r) = r + 1$$

$$\begin{aligned} P(D)\,Q(D)\,y &= P(D)\left[Q(D)\,y\right] \\ &= P(D)\left[(D+1)y\right] \\ &= P(D)\left[Dy + y\right] \\ &= P(D)\left[y' + y\right] \end{aligned}$$

$$= (D^2+1)(y'+y)$$

$$= D^2(y'+y) + (y'+y)$$

$$= D^2y' + D^2y + y' + y$$

$$P(D)Q(D)y = y^{(3)} + y^{(2)} + y' + y$$

$$Q(D)\,P(D)y = Q(D)\left[P(D)y\right]$$

$$= Q(D)\left[(D^2+1)y\right]$$

$$= Q(D)\left[D^2y + y\right]$$

$$= Q(D)\left[y'' + y\right]$$

$$= (D+1)(y'' + y)$$

$$= (D+1)y'' + (D+1)y$$

$$= (Dy'' + y'') + (Dy + y)$$

$$Q(D)P(D)y = y''' + y'' + y' + y$$

$$P(D)Q(D)y = y^{(3)} + y^{(2)} + y' + y$$

$$P(D)Q(D)\,y = Q(D)P(D)\,y$$

$$\Rightarrow P(D)Q(D) = Q(D)P(D)$$

Example 8.2.3 If $P(D) = D - 5$ and $Q(D) = D + 7$, verify that

$$P(D)Q(D) = Q(D)P(D).$$

$$\begin{aligned}
P(D)Q(D)(y) &= (D-5)(D+7)\,y \\
&= (D-5)(Dy + 7y) \\
&= (D-5)(y' + 7y) \\
&= Dy' + 7Dy - 5y' - 35y \\
&= y'' + 7y' - 5y' - 35y \\
&= y'' + 2y' - 35y
\end{aligned}$$

$$Q(D)P(D)\,y = (D+7)(D-5)\,y$$
$$= (D+7)(Dy-5y)$$
$$= (D+7)(y'-5y)$$
$$= Dy' - 5Dy + 7y' - 35y$$
$$= y'' - 5y' + 7y' - 35y$$
$$= y'' + 2y' - 35y$$
$$\therefore\ P(D)Q(D) = Q(D)P(D)$$

Theorem 8.2.4 If $P(D) = P_1(D)P_2(D)\cdots P_k(D)$, where each $P_i(D)$ is a polynomial differential operator, then, for each i, $1 \le i \le k$, any solution to $P_i(D)y = 0$ is also a solution to $P(D)y = 0$.

PF. SUPPOSE y SOLVES $P_i(D)y=0$

$$P(D)y = \underbrace{[P_1(D) \cdots P_k(D)]}_{\text{REARRANGE (COMMUTATIVE)}} y$$

$$= [P_1 P_2 \cdots P_{i-1} P_{i+1} \cdots P_k] \underbrace{P_i(D)y}_{0}$$

$$= 0$$

$\Rightarrow$ y SOLVES $P(D)y=0$. ∎

Example 8.2.8 Determine the general solution to $y'' - y' - 2y = 0$.

$$D^2 y - Dy - 2y = (D^2 - D - 2)y$$

$$P(r) = r^2 - r - 2 = (r-2)(r+1)$$

$$P(D)y = 0$$

$$P(r) = P_1(r)\, P_2(r)$$

$$P_1(r) = r - 2$$

$$P_2(r) = r + 1$$

$$\boxed{P_1(D)y = 0} \quad \& \quad P_2(D)y = 0$$

$$P_1(D)y = 0 \Rightarrow (D-2)\,y = 0$$

$$Dy - 2y = 0 \Rightarrow y' - 2y = 0$$

$$\Rightarrow y = e^{2x}$$ [SEPARABLE EQUATION]

$$P_2(D)y = 0 \Rightarrow (D+1)\,y = 0$$

$$\Rightarrow Dy + y = 0 \Rightarrow y' + y = 0$$

$$\Rightarrow y = e^{-x}$$

$$P_1(D)y = 0 \qquad\qquad P_2(D)y = 0$$

$$y_1 = e^{2x} \qquad\qquad y_2 = e^{-x}$$

y_1 & y_2 BOTH SOLVE $P(D)y = 0$

$$W[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix}$$

$$= (e^{2x})(-e^{-x}) - (2e^{2x})(e^{-x})$$

$$= -3e^{x} \neq 0$$

y_1 & y_2 ARE L.I.

$\Rightarrow$ GENERAL SOLN.

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 e^{2x} + c_2 e^{-x}$$

Example 8.2.9 Determine the general solution to $y'' + 6y' + 25y = 0$.

$$D^2 y + 6Dy + 25y = 0$$

$$\Rightarrow (D^2 + 6D + 25)y = 0$$

$$P(r) = r^2 + 6r + 25$$

DISCRIMINANT $= b^2 - 4ac$

$= 6^2 - (4)(1)(25)$

$= 36 - 100$

$= -64$

QUADRATIC FORMULA $\longrightarrow$ $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-6 \pm \sqrt{-64}}{2}$$

NOTE: SEE LATER FOR WHAT i MEANS

$\sqrt{-64} = \sqrt{-1} \cdot \sqrt{64}$

$= i8$

$$r = \frac{-6 \pm 8i}{2} = -\frac{6}{2} \pm \frac{8i}{2} = -3 \pm 4i$$

$$r_1 = -3 + 4i \quad , \quad r_2 = -3 - 4i$$

$$(r^2 + 6r + 25) = \underbrace{(r - r_1)}_{P_1} \underbrace{(r - r_2)}_{P_2}$$

$$P(D)\, y = 0 \longleftrightarrow P_1(D)\, y = 0$$

$$\& \quad P_2(D)\, y = 0$$

NOTE: SEE LATER FOR WHAT i MEANS

$P_1(D)\,y$

$(D - r_1)\,y = y' - (-3+4i)\,y = 0$

$\Rightarrow \int \frac{dy}{y} = \int (-3+4i)\,dx$

$\Rightarrow \ln y = (-3+4i)\,x$

$\Rightarrow y = e^{(-3+4i)x}$

NOTE: SEE LATER FOR WHAT i & e^{ix} MEAN

$y = e^{-3x} \cdot e^{i4x} = e^{-3x}(\cos 4x + i \sin 4x)$

$$P_2(D)\,y = 0$$

$$(D - \lambda_2)\,y = 0 \quad \Rightarrow \quad y' - (-3-4i)\,y = 0$$

$$\Rightarrow \frac{dy}{y} = (-3-4i)\,dx$$

$$\Rightarrow y = e^{(-3-4i)x}$$

$$= e^{-3x}\left[\cos(-4x) + i\sin(-4x)\right]$$

$$= e^{-3x}\left[\cos 4x - i\sin 4x\right]$$

NOTE: SEE LATER FOR WHAT i & e^{ix} MEAN

$$y_1^{\#} = e^{-3x}\cos 4x + i e^{-3x}\sin 4x$$

$$y_2^{\#} = e^{-3x}\cos 4x - i e^{-3x}\sin 4x$$

NOTE: SEE LATER FOR WHAT i & e^{ix} MEAN

$$y_1 = \frac{y_1^{\#} + y_2^{\#}}{2} = e^{-3x}\cos 4x$$

$$y_2 = \frac{y_1^{\#} - y_2^{\#}}{2i} = e^{-3x}\sin 4x$$

CHECK:

$W[y_1, y_2] \neq 0$

y_1 & y_2 SOLVE $y'' + 6y' + 25y = 0$

BREAK
TILL
10:40

DETOUR : COMPLEX NUMBERS.

Q. WHAT HAPPENS WHEN $p(x) = ax^2 + bx + c$, $(b^2 - 4ac < 0)$?

$$x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \sqrt{-1}$$

$$i = \sqrt{-1}$$

$$p(x) = ax^2 + bx + c$$

$$= ax^2 + bx + \frac{b^2}{4a} + c - \frac{b^2}{4a}$$

$$= a\underbrace{\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right]}_{\left(x+\frac{b}{2a}\right)^2} + \underbrace{\left[c - \frac{b^2}{4a}\right]}_{-\frac{D}{4a}}$$

$(D = b^2 - 4ac)$

$$p(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

$p(x) = 0$

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a} = 0 \Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}$$

$$\left(x + \frac{b}{2a}\right) = \pm \sqrt{\frac{D}{4a^2}} = \pm \frac{\sqrt{D}}{2a}$$

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}, \quad i^2 = -1$

$i = \sqrt{-1} \Leftrightarrow i^2 = -1$

$(a, b, c, d \in \mathbb{R})$

<u>DEFN</u>: $(a+bi) + (c+di) = (a+c) + i(b+d)$

$(a+bi)(c+di) = (a+bi)c + (a+bi)di$

$= ac + ibc + iad + i^2 bd$

$= ac + ibc + iad - bd$

$= (ac - bd) + i(bc + ad)$

<u>FACT</u>: $(\mathbb{C}, +, \times)$ IS A FIELD.

CAN DO LINEAR ALGEBRA OVER IT.

COMPLEX EXPONENTIAL

$$e^{x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{i^n \cdot x^n}{n!}$$

$\boxed{\text{SEE LATER FOR HOW TO COMPUTE } i^n}$

$$= 1 + \frac{ix}{1!} - \frac{1 \cdot x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!}$$

$$- x^6/6! - ix^7/7! + \cdots$$

$$= \overbrace{\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \right]}^{\cos x}$$

$$+ \; i \underbrace{\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right]}_{\sin x}$$

$$\boxed{e^{ix} = \cos x + i \sin x}$$

EULER'S FORMULA.

$i^0 = 1$

$i^1 = i$

$i^2 = -1$

$i^3 = i^2 \cdot i = -i$

$i^4 = i^3 \cdot i = (-i)(i) = -i^2 = -(-1) = 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

$i^6 = i^5 \cdot i = i \cdot i = i^2 = -1$

i^0	1
i^1	i
i^2	-1
i^3	$-i$
i^4	1
i^5	i
i^6	-1
i^7	$-i$
i^8	1
i^9	i

$i^{4n} = 1$

$i^{4n+1} = i^{4n} \cdot i = i$

$i^{4n+2} = i^{4n} \cdot i^2 = i^2 = -1$

$i^{4n+3} = i^{4n} \cdot i^3 = i^3 = -i$

$$e^{x+iy} = e^x \cdot e^{iy}$$

$$= e^x \left[\cos y + i \sin y \right]$$

$\boxed{\text{EULER'S FORMULA} \Rightarrow \text{DOUBLE-ANGLE FORMULAE}}$

$$e^{i2x} = \cos 2x + i \sin 2x$$

$$e^{i2x} = \left(e^{ix}\right)^2 = \left(\cos x + i \sin x\right)^2$$

$$= (\cos x)^2 + 2(\cos x)(i \sin x) + (i \sin x)^2$$

$$\left(\cos 2x + i \sin 2x\right) = \left(\cos^2 x - \sin^2 x\right) + i\left(2 \cos x \sin x\right)$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\sin 2x = 2 \sin x \cos x$$

Lemma 8.2.5 Consider the differential operator $(D-r)^m$, where m is a positive integer, and r is a real or complex number. For any $u \in C^m(I)$,

$$(D-r)^m(e^{rx}u) = e^{rx}D^m(u). \qquad (8.2.4)$$

$m=0$,

$$(D-r)^0(e^{rx}u) = e^{rx}D^0u$$

$$e^{rx}u = e^{rx}u$$

$m=1$

$$(D-r)(e^{rx}u) = D(e^{rx}u) - re^{rx}u$$

$$= \frac{d}{dx}(e^{rx}u) - re^{rx}u = \cancel{re^{rx}u} + e^{rx}u' \cancel{- re^{rx}u}$$

$$= e^{\lambda x} u' = e^{\lambda x} D u$$

$$(D-\lambda)^{m+1}(e^{\lambda x} u) = (D-\lambda)(D-\lambda)^{m}[e^{\lambda x} u]$$

$$= (D-\lambda)\left[e^{\lambda x} \underbrace{D^{m} u}\right]$$

$$= (D-\lambda)(e^{\lambda x} v)$$

$$= e^{\lambda x} D v = e^{\lambda x} D(D^{m} u)$$

$$= e^{\lambda x} D^{m+1}(u)$$

Theorem 8.2.6 The differential equation $(D-r)^m y = 0$, where m is a positive integer and r is a real or complex number, has the following m solutions that are linearly independent on any interval:

$$e^{rx}, xe^{rx}, x^2e^{rx}, \ldots, x^{m-1}e^{rx}.$$

$$y = e^{rx} u$$

$$0 = (D-r)^m y = (D-r)^m (e^{rx} u)$$

$$= e^{rx} D^m u$$

$$\therefore \underbrace{e^{rx}}_{\neq 0} D^m u = 0 \rightarrow D^m u = 0$$

$$D^m u = 0 \to \frac{d^m}{dx^m} u = 0$$

$$D^{m-1} u = c_0$$

$$D^{m-2} u = c_0 x + c_1$$

$$D^{m-3} u = \frac{c_0}{2} x^2 + c_1 x + c_2$$

$$\vdots$$

$D^{m-k} u =$ POLYNOMIAL OF DEGREE $(k-1)$

u = POLYNOMIAL OF DEGREE $m-1$.

$$P_{m-1}(\mathbb{R}) = \text{span}\{1, x, x^2, \dots, x^{m-1}\}$$

$$\boxed{u_j = x^{j-1} \qquad 1 \le j \le m}$$

$$y_j = e^{\lambda x} u_j = x^{j-1} e^{\lambda x} \quad (1 \le j \le m)$$