MATH 165 (SUMMER '22, SESH B2)

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TA: PABLO BHOWMIK
OFF HRS:
M, F 3:00 PM - 4:00 PM (ET)
STARTING THIS FRIDAY

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEBPAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURE 1 ARE UPLOADED. LMK IF UNAVAILABLE
2. WW 00 – DUE TODAY (28th JUNE) AT 11:00 PM ET
   WW 01 – DUE SATURDAY (2nd JULY) AT 11:0 PM ET
3. OFFICE HOURS ANNOUNCED (SEE PREVIOUS PAGE)
4. ACADEMIC HONESTY QUIZ → COMPLETE ON BLACKBOARD
5. CONFLICTS WITH EXAM TIMES ?
6. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE !

§ 1.2 BASIC IDEAS AND TERMINOLOGY (CONT'D.)

REMINDER:

LINEAR $\approx$ DIFF. EQN. DEPENDS LINEARLY ON $y, y', y'', \ldots, y^{(n)}$

NONLINEAR $\approx$ ANYTHING ELSE

N.B.: A TERM LIKE yy' IS NONLINEAR BUT A TERM LIKE x^2 IS LINEAR.

REMINDER:

Def (SOLUTION TO AN ODE)

DEFINITION 1.2.4

A function $y = f(x)$ that is (at least) n times differentiable on an interval I is called a **solution** to the differential equation (1.2.1) on I if the substitution $y = f(x)$, $y' = f'(x), \ldots, y^{(n)} = f^{(n)}(x)$ reduces the differential equation (1.2.1) to an identity valid for all x in I. In this case we say that $y = f(x)$ **satisfies** the differential equation.

e.g. $y = e^x$ FOR $y' = y$ $\qquad (e^x)' = e^x$

e.g. $y = \sin x$ FOR $y'' + y = 0$ $\qquad (\sin x)'' = -\sin x$

Def : GENERAL SOLN. ON I

DEFINITION 1.2.8

A solution to an nth-order differential equation on an interval I is called the **general solution on** I if it satisfies the following conditions:

1. The solution contains n constants $\boxed{c_1, c_2, \ldots, c_n.}$ $\rightarrow$ PARAMETERS.
2. All solutions to the differential equation can be obtained by assigning appropriate values to the constants.

e.g., $y' = y \quad \longleftrightarrow \quad y = ce^{x}$ $\qquad c \rightarrow$ PARAMETER

$$\frac{d}{dx} \sin 3x = 3 \cos 3x$$

Example 1.2.9 Determine the general solution to the differential equation $y'' = 18\cos 3x$.

$$\frac{d}{dx} \cos 3x = -3 \sin 3x$$

$$y'' = 18 \cos 3x$$

INTEGRATE

DIFF

$$y' = \int (18 \cos 3x)\, dx$$

$$y' = 6 \sin 3x + c_1$$

INTEGRATE

DIFF.

$$y = \int (6 \sin 3x + c_1)\, dx = -2 \cos 3x + c_1 x + c_2$$

$$y'' = 18 \cos 3x \iff \boxed{y = -2 \cos 3x + c_1 x + c_2}$$

GENERAL SOLN

$G(x, y, y', \dots, y^{(n)}) = 0$

DEFINITION 1.2.10

An nth-order differential equation together with n auxiliary conditions of the form

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \dots, y_{n-1}$ are constants, is called an **initial-value problem**.

$x_0, y_0, \dots, y_{n-1} \in \mathbb{R}$

Example 1.2.11 Solve the initial-value problem

$$\boxed{y'' = 18\cos 3x}$$

$$y(0) = 1, \qquad y'(0) = 4.$$

$c_2 = 3$ $\qquad$ $c_1 = 4$

$$y = -2\cos 3x + c_1 x + c_2$$

$$1 = y(0) = -2\cos(3\times 0) + c_1 \times 0 + c_2$$

$$= -2 + c_2 \rightarrow c_2 = 2 + 1 = 3$$

$$y' = \frac{d}{dx}\left(-2\cos 3x + c_1 x + c_2\right) = 2\sin 3x + c_1$$

$$4 = y'(0) = 2\sin(3\times 0) + c_1 = c_1 \Rightarrow c_1 = 4$$

$$\boxed{y = -2\ln 3x + 4x + 3} \rightarrow$$

SOLN. OF THE I.V.P.

EXISTENCE & UNIQUENESS

IVP : ODE + INITIAL VALUE DATA — (I)

XED)

Q1 (EXISTENCE) DOES (I) HAVE A SOLUTION?

Q2 (UNIQUENESS) IF YES, DOES (I) HAVE ONLY <u>ONE</u> SOLN?

(WHY? → REAL-WORLD APPLICATIONS!)

UNIQUENESS THEOREM → n^{th} ORDER LINEAR ODEs ↑ IVP ALWAYS HAVE EXISTENCE & UNIQUENESS

Theorem 1.2.12 Let $a_1, a_2, \ldots, a_n, F$ be functions that are continuous on an interval I. Then, for any x_0 in I, the initial-value problem

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)$$

n^{th} ORDER LINEAR ODE

$$\boxed{y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}}$$

has a unique solution on I.

INITIAL VALUE DATA

Example 1.2.13 Prove that the general solution to the differential equation

$$y'' + \omega^2 y = 0, \quad -\infty < x < \infty,$$

where ω is a nonzero constant, is

$$y(x) = c_1 \cos \omega x + c_2 \sin \omega x,$$ $\to y''$

where c_1, c_2 are arbitrary constants.

$$y' = \frac{d}{dx}\left(c_1 \cos \omega x + c_2 \sin \omega x\right)$$

$$= c_1(-\omega \sin \omega x) + c_2(\omega \cos \omega x)$$

$$y'' = \frac{d}{dx}\left[-\omega c_1 \sin \omega x + \omega c_2 \cos \omega x\right] = \left[-\omega^2 c_1 \cos \omega x - \omega^2 c_2 \sin \omega x\right]$$

$$= -\omega^2 y$$

$$y'' = -\omega^2 y \iff y'' + \omega^2 y = 0$$

$y_1 \rightsquigarrow$ SOLVES THE O.D.E.

$$y_2(x) = y_1(0) \cos \omega x + \frac{y_1'(0)}{\omega} \sin \omega x$$

$$c_1 = y_1(0) \quad , \quad c_2 = \frac{y_1'(0)}{\omega}$$

$$y_2(0) = y_1(0) \underbrace{\cancel{\cos(\omega \cdot 0)}}_{1} + \cancel{\frac{y_1'(0)}{\omega} \sin(\omega \cdot 0)} = y_1(0)$$

$$y_2' = \frac{d}{dx}\left(y_1'(0)\cos\omega x + \frac{y_1'(0)}{\omega}\sin\omega x \right)$$

$$= -\omega y_1'(0)\sin\omega x + y_1'(0)\cos\omega x$$

$$y_2'(0) = \cancel{-\omega\, y_1'(0)\sin(\omega\cdot 0)} + y_1'(0)\underbrace{\cancel{\cos(\omega\cdot 0)}}_{1}$$

$$= y_1'(0)$$

$$\boxed{\begin{array}{l} y_1(0) = y_2(0) \\ y_1'(0) = y_2'(0) \end{array}} \Longrightarrow y_1(x) = y_2(x) \quad \text{EVERY WHERE}$$

$$y_1 = y_2$$

§ 1.3 THE GEOMETRY OF FIRST ORDER DIFF. EQNS.

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

1st ORDER ODE : $\dfrac{dy}{dx} = f(x,y)$

e.g. $\dfrac{dy}{dx} = 1$, $\dfrac{dy}{dx} = y - x$, $\dfrac{dy}{dx} = x$, $\dfrac{dy}{dx} = e^y$

NOTE 1 : $dy/dx \rightsquigarrow$ SLOPE OF THE TANGENT

NOTE 2 : GENERAL SOLN. IS A ONE-PARAMETER FAMILY OF CURVES.

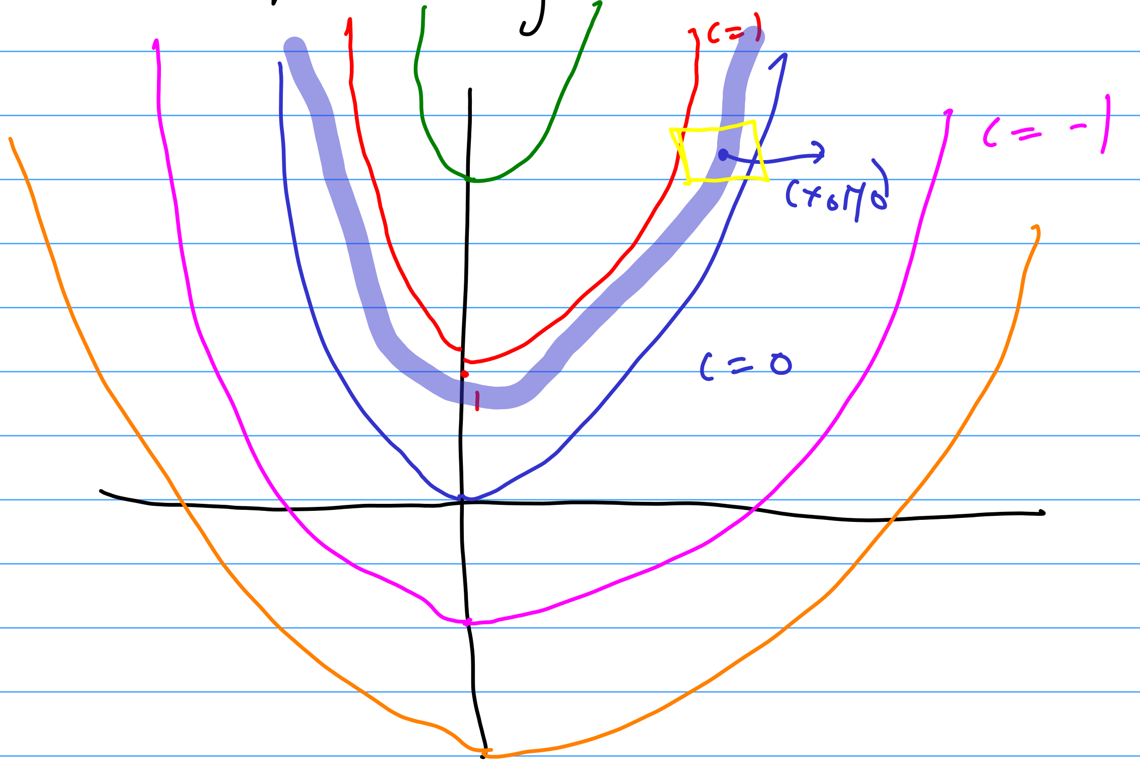
$y = g(x)$ IS A SOLN

$$\frac{dy}{dx} = f(x, y)$$

Example 1.3.1 Find the general solution to the differential equation $dy/dx = 2x$, and sketch the corresponding solution curves.

$$\frac{dy}{dx} = 2x \quad \Rightarrow \quad y = \int 2x\,dx = x^2 + c$$

$$y = x^2 + c$$



BREAK TILL

10:07 AM ET

PARTIAL DERIVATIVES

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$f(x,y)$$

$$\frac{\partial f}{\partial y} \rightsquigarrow$$ PRETEND THAT x IS CONSTANT & DIFFERENTIATE W.R.T. y.

$$f(x,y) = x - y \rightsquigarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x - y) = -1$$

$$f(x,y) = x y^2 \rightsquigarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x y^2\right) = x(2y) = 2xy$$

Theorem 1.3.2 **(Existence and Uniqueness Theorem)**

Let $f(x, y)$ be a function that is continuous on the rectangle

$$\frac{dy}{dx} = f(x,y)$$

$$R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}.$$

Suppose further that $\frac{\partial f}{\partial y}$ is continuous in R. Then for any interior point (x_0, y_0) in the rectangle R, there exists an interval I containing x_0 such that the initial-value problem (1.3.2) has a unique solution for x in I.

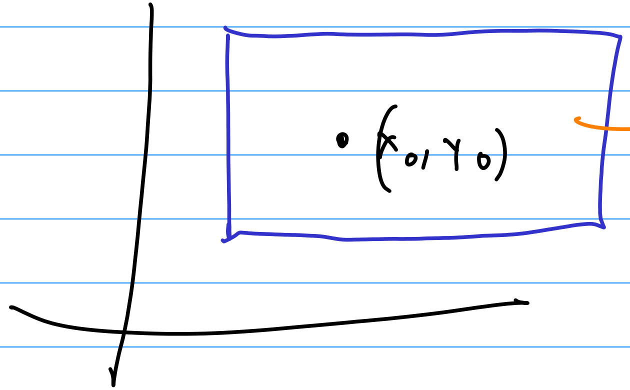
N.B.: SOLN. CURVES OF $dy/dx = f(x,y)$ CANNOT INTERSECT!

IVP :

$$\frac{dy}{dx} = f(x,y)$$

$$y(x_0) = y_0$$

$(x_0, y_0 \in \mathbb{R}, \text{ FIXED})$



$\longrightarrow f, \ \frac{\partial f}{\partial y}$

BOTH CONTINUOUS ON THIS RECTANGLE

(RETURN TO PAST EX.)

$$\frac{dy}{dx} = 2x$$

$$f(x,y) = 2x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2x) = 0$$

$$\frac{d}{dx} x^m = m x^{m-1}$$

Example 1.3.3 Prove that the initial-value problem

$x_0 = 0$

$y_0 = a \neq 0$

$$\frac{dy}{dx} = 3xy^{1/3}, \qquad y(0) = a$$

has a unique solution whenever $a \neq 0$.

$$f(x,y) = \underbrace{3xy^{1/3}}_{\text{CONTINUOUS EVERYWHERE}} \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(3xy^{1/3}\right)$$

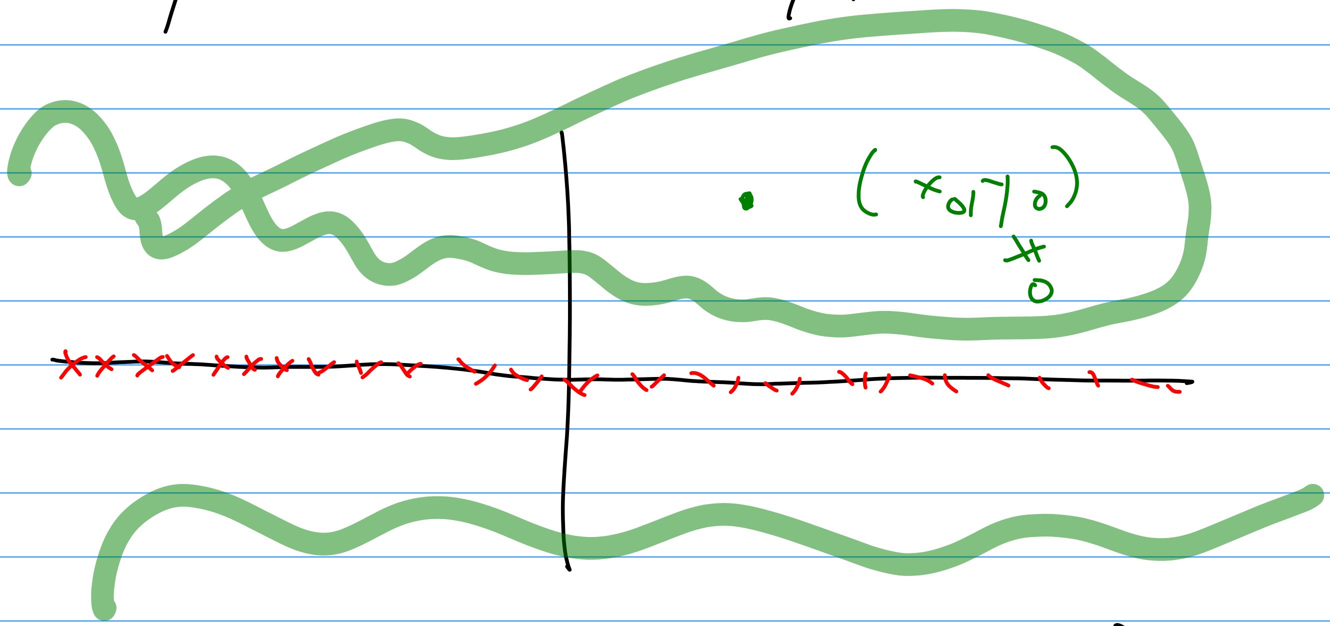
$$= 3x \frac{\partial (y^{1/3})}{\partial y} = 3x\left(\frac{1}{3} y^{-2/3}\right)$$

$$\frac{\partial f}{\partial y} = xy^{-2/3}$$

NOT CONT. AT

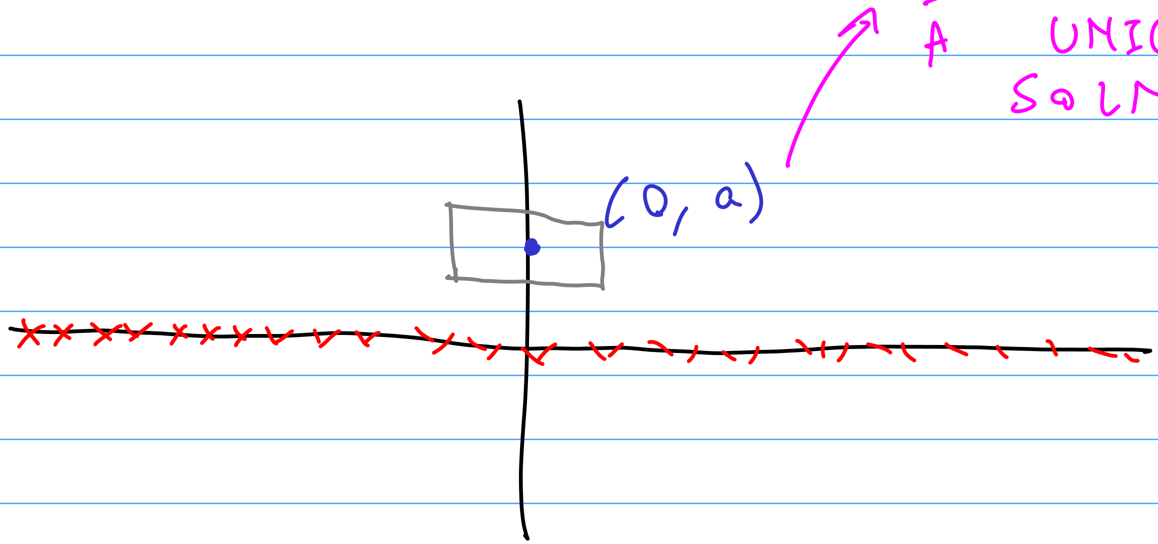
$y = 0$

$$\frac{\partial f}{\partial y} \to \text{IS CONTINUOUS IF } y \neq 0$$



$\neq 0$

$(x_0, y_0) = (x, a)$



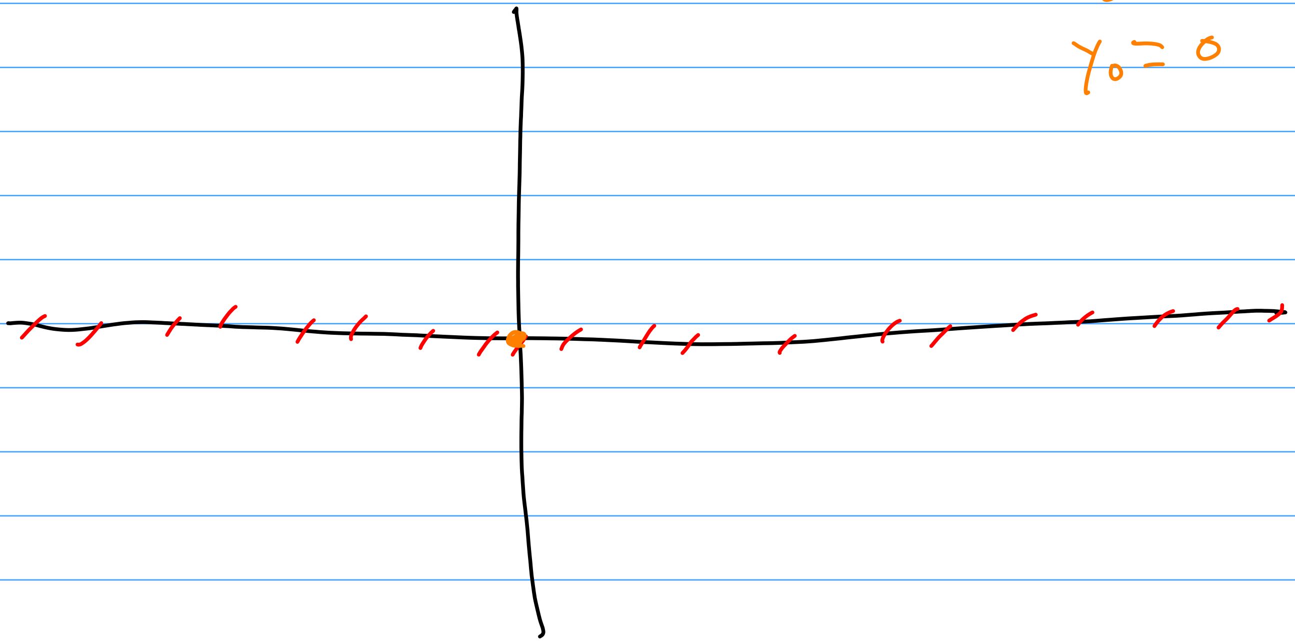
I.V.P. HAS A UNIQUE SOLN.

Example 1.3.4 Discuss the existence and uniqueness of solutions to the initial-value problem

$$\boxed{\frac{dy}{dx} = 3xy^{1/3}, \qquad y(0) = 0.} \quad 1$$

$x_0 = 0$

$y_0 = 0$



$$\frac{dy}{dx} = 3xy^{1/3} = f(x,y)$$

$$y_1(x) \equiv 0$$

$$y_1'(x) = 0$$

(PLUG y_1 INTO I) $\quad (0) = (3x)(0^{1/3})$

$\Rightarrow$ $y_1 \equiv 0$ IS A SOLN TO I

$$\boxed{\frac{dy}{dx} = 3x\,y^{1/3}}$$

$$y_2(x) = x^3$$

$$y_2(0) = 0^3 = 0$$

$$y_2'(x) = 3x^2$$

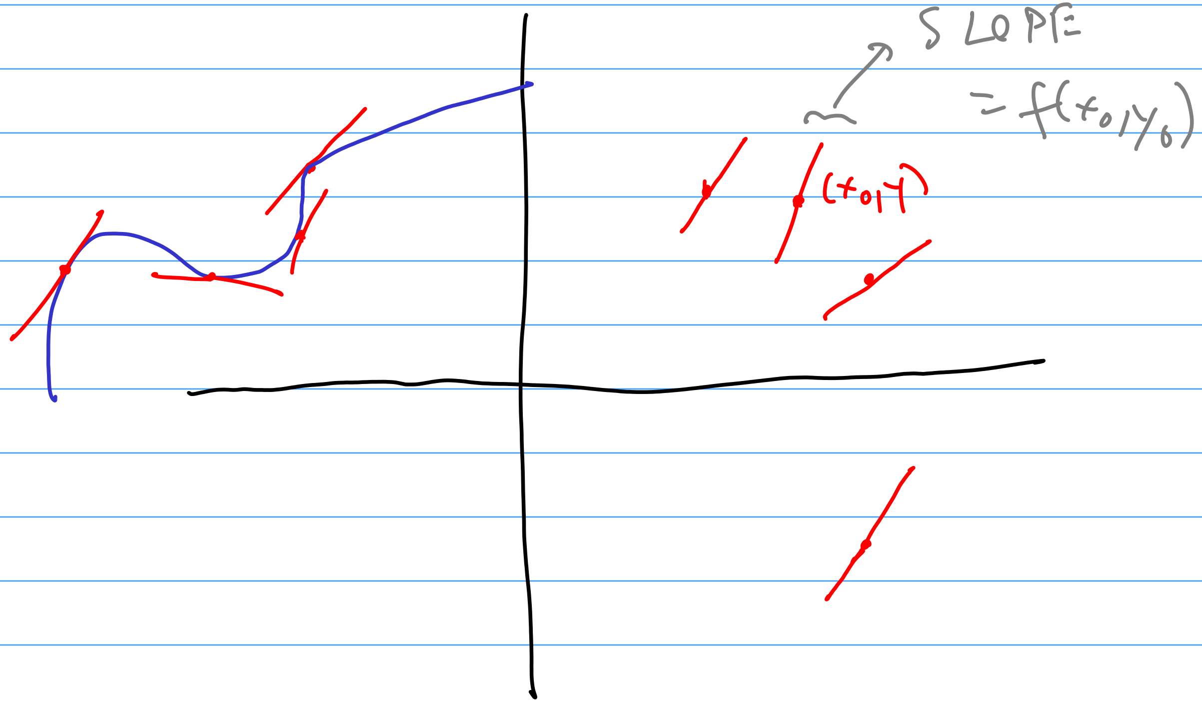
$\downarrow$ PLUG IT IN.

$$3x^2 = (3x)\left(x^3\right)^{1/3} = (3x)(x) = 3x^2$$

$\Rightarrow$ y_2 IS ALSO A SOLN. TO I.V.P. I

SLOPE FIELDS

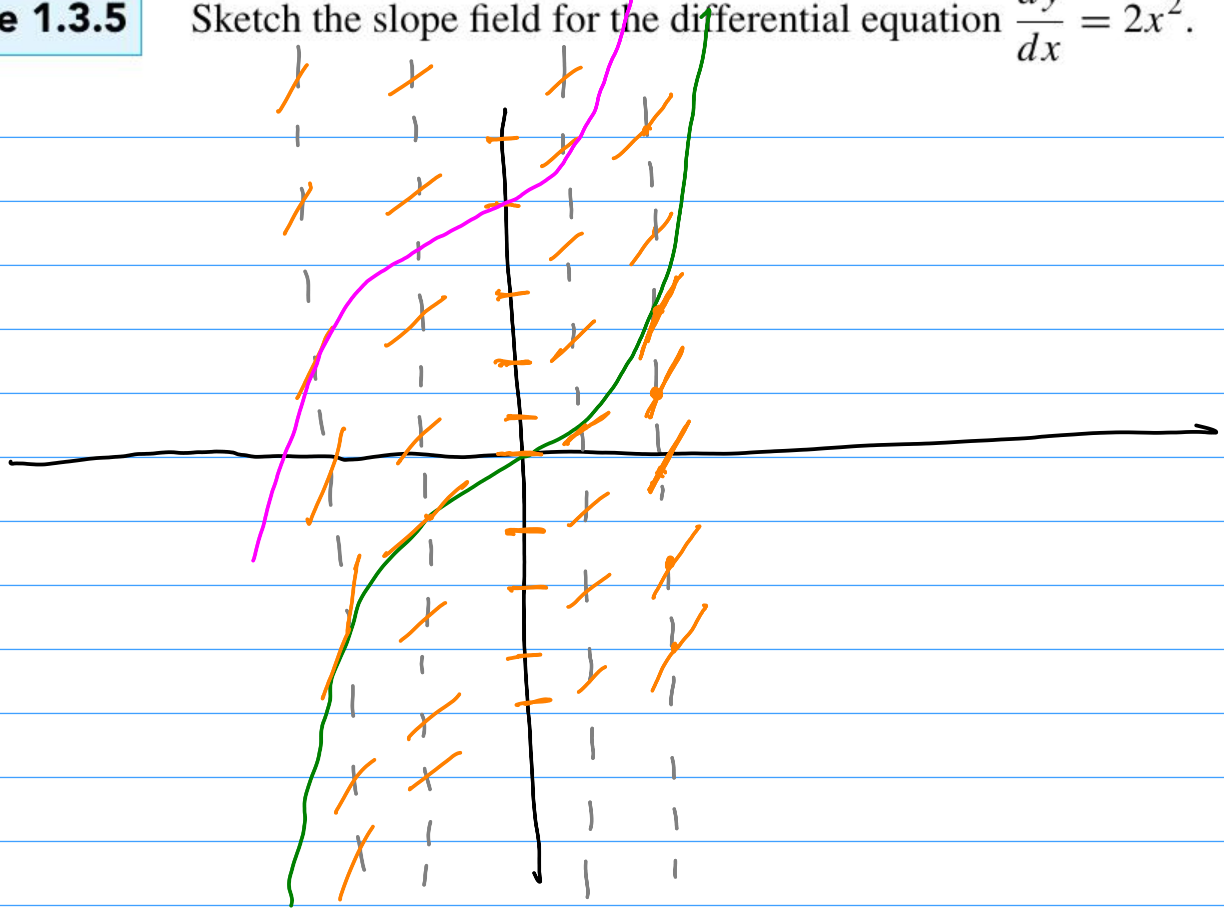
$$\frac{dy}{dx} = f(x, y)$$



$y = x^3 + 5$

$y = x^3$

Example 1.3.5 Sketch the slope field for the differential equation $\dfrac{dy}{dx} = 2x^2$.



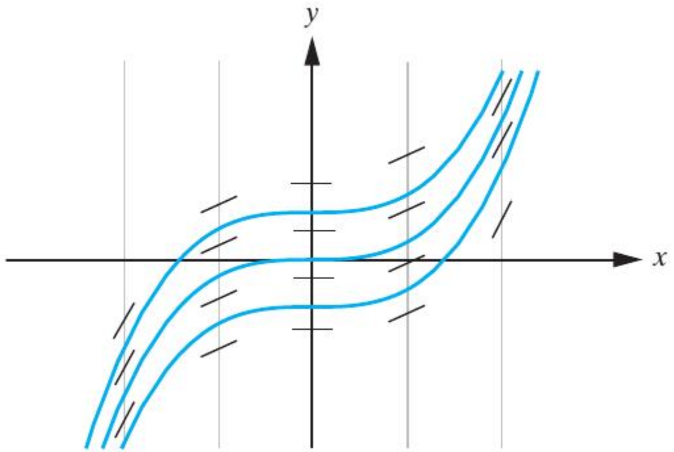


Figure 1.3.5: Slope field and some representative solution curves for the differential equation $\dfrac{dy}{dx} = 2x^2$.

$k \in \mathbb{R}$, FIXED

1. ISOCLINES $\quad f(x,y) = k \longrightarrow$ CURVE GIVEN BY THIS EQUATION

2. EQUILIBRIUM SOLUTION
(A SOLUTION OF THE FORM $y = \text{CONSTANT}$)

(SOLN. CURVES CANNOT CROSS EQUILIBRIA)

EQ. SOLNS.

$y(x) = c$ SOLVES $\dfrac{dy}{dx} = f(x, y)$

$y'(x) = 0$

$0 = f(x, c)$

IDENTITY

$$\frac{dy}{dx} = \boxed{(y-x)(y-1)}$$

$\cancel{y = x}$ $\boxed{y=1}$

$$0 = \left(\frac{d}{dx}(1)\right) = (1-x)(1-1) = 0$$

$y=1$ IS A SOLN.!

3. CONCAVITY (POSTPONED TO NEXT CLASS)

Example 1.3.6 Sketch the slope field and some approximate solution curves for the differential equation

$$\frac{dy}{dx} = y(2-y). = f(x,y) \tag{1.3.4}$$

ISOCLINES

$$f(x,y) = k$$

$$\Updownarrow$$

$$y(2-y) = k$$

$$\Rightarrow \quad y^2 - 2y + k = 0$$

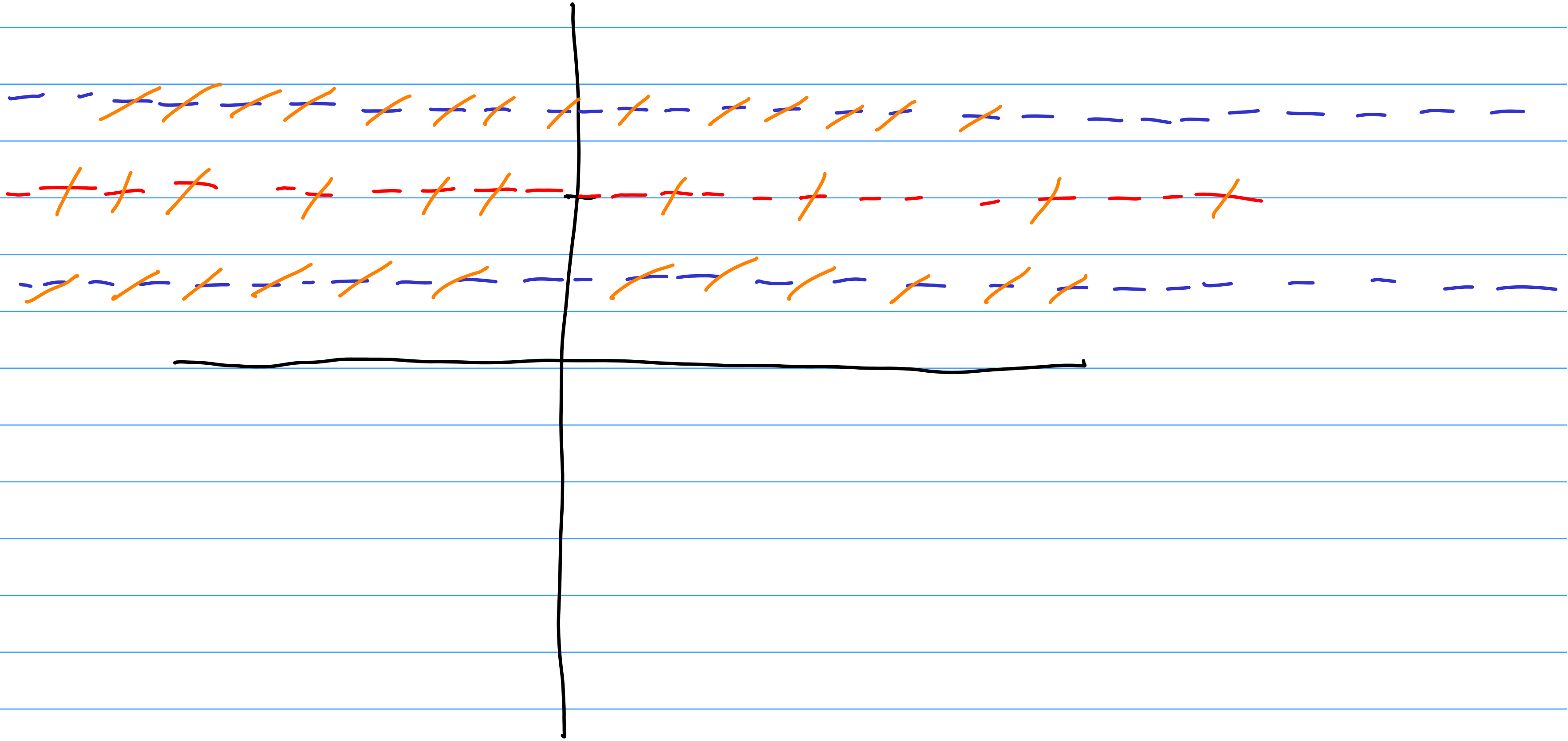
$$\Rightarrow y = \frac{2 \pm \sqrt{4-4k^2}}{2} = 1 \pm \sqrt{1-k^2}$$

$$f(x,y) = y(2-y)$$

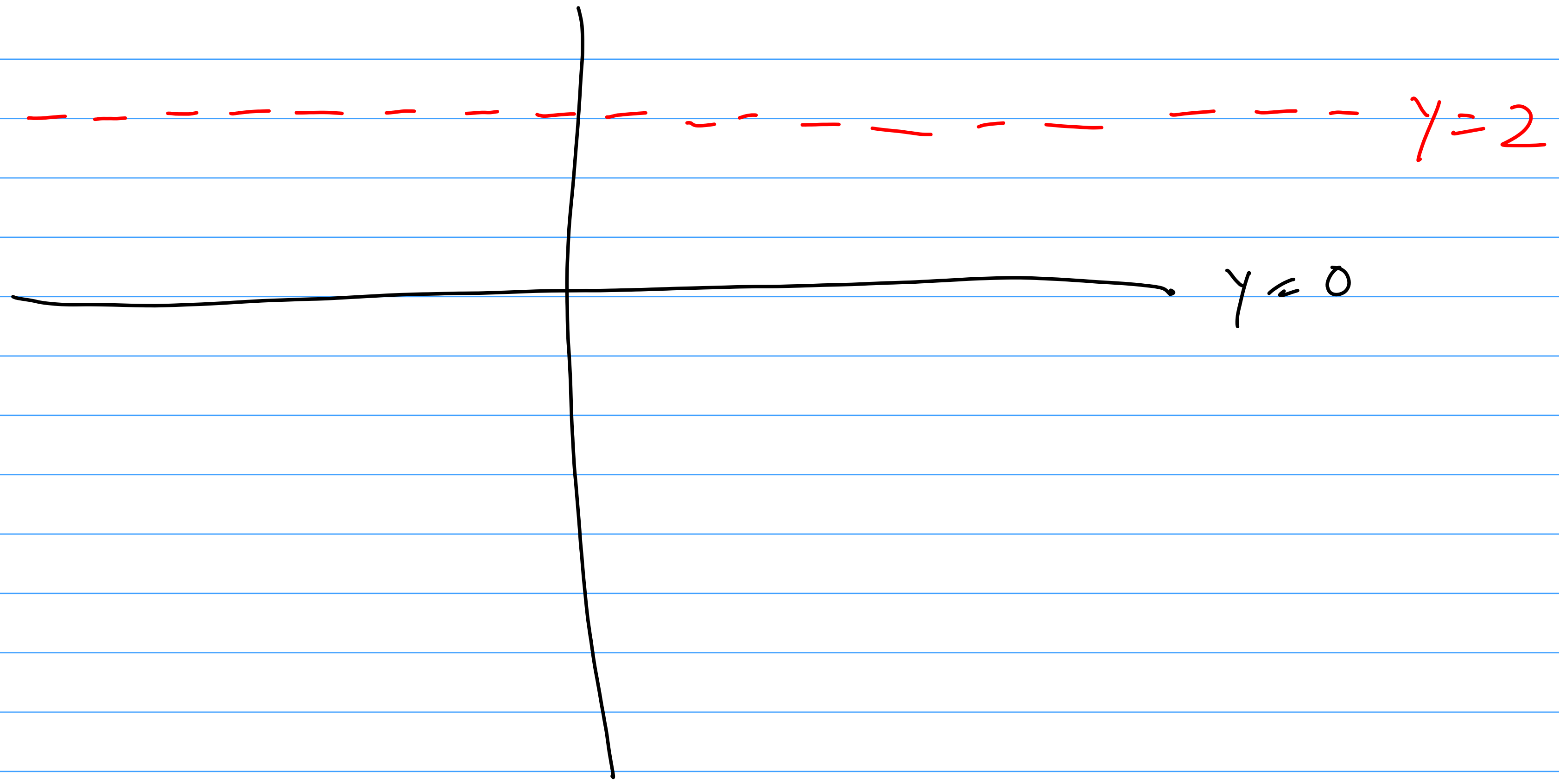
EQ. SOLNS: $y = 0$ & $y = 2$

$f(x,y) = 1/2$

$f(x,y) = 1$



EQ SOLTNS



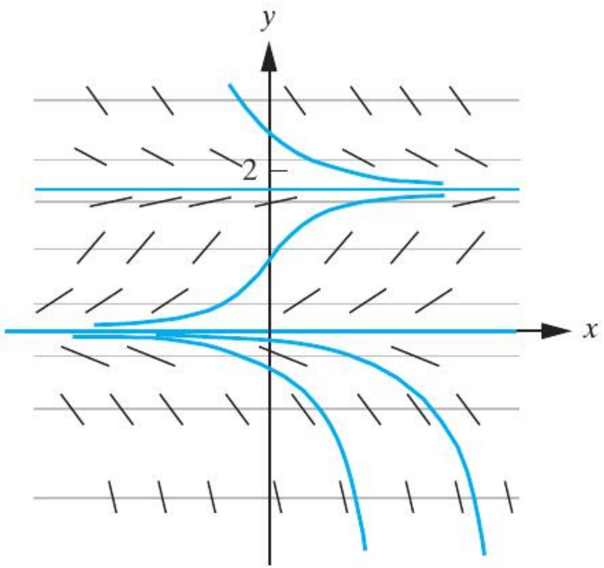


Figure 1.3.6: Hand-drawn slope field, isoclines, and some solution curves for the differential equation $\frac{dy}{dx} = y(2 - y)$.

$$y(x) = \frac{2ce^{2x}}{ce^{2x} - 1}$$

Example 1.3.7 Sketch the slope field for the differential equation

$$\frac{dy}{dx} = y - x.$$

ISOCLINES: $f(x,y) = y - x = k$

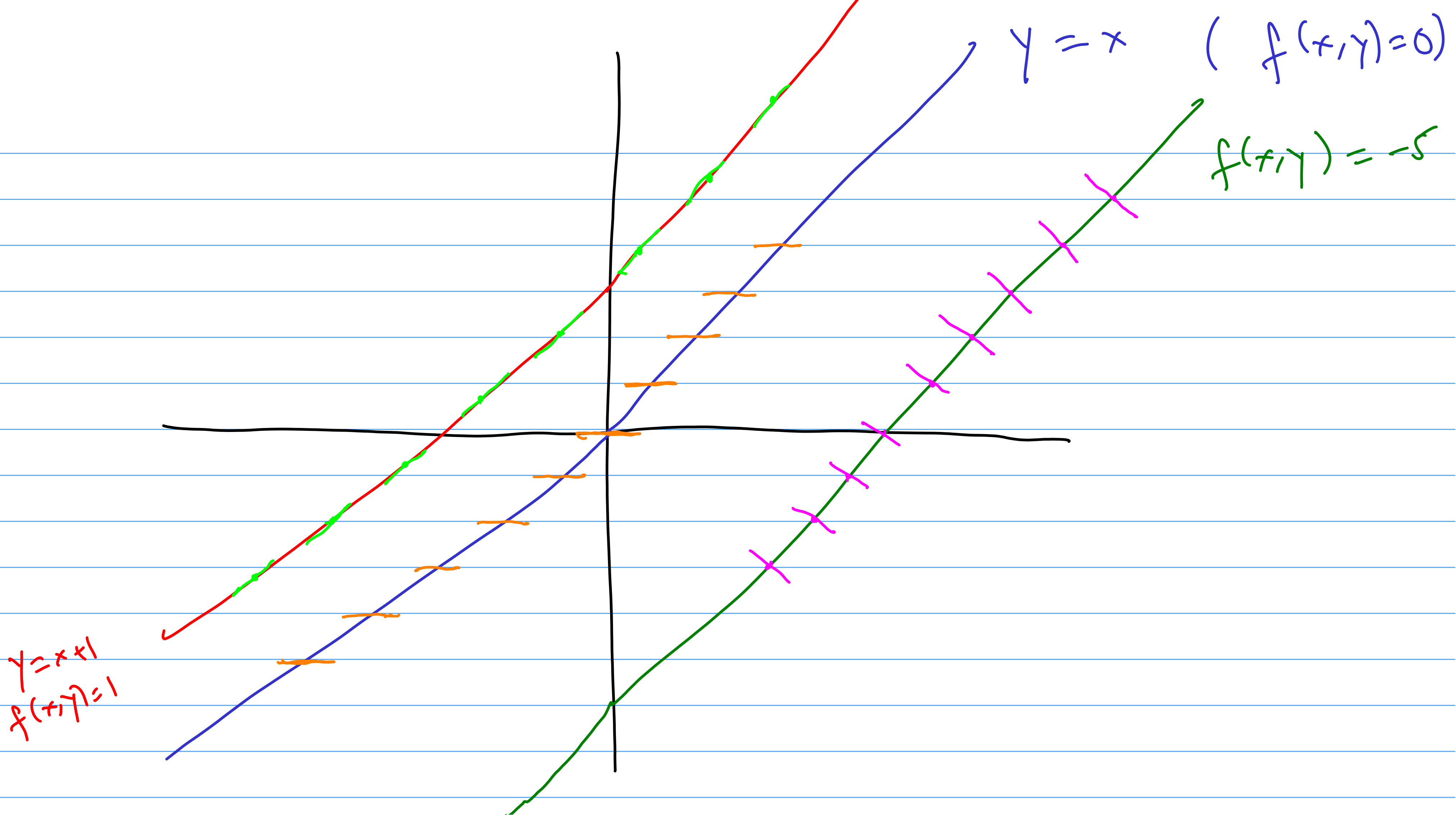
$\boxed{y = x + k} \longleftrightarrow f(x,y) = k$

EQUILIBRIUM SOLNS: $f(x,y) = \boxed{y - x}$

NONE!

$c - x \neq 0$
$\forall c$

$y = x$ IS $\boxed{NO?}$ AN EQ. SOLN!



y = x (f(x,y)=0)
f(x,y) = -5
y=x+1
f(x,y)=1