MATH 165 (SUMMER '22, SESH B2)

OFFICE HOURS (WEEK 6):

PABLO :

M – 2:00 PM - 3:00 PM (ET)

T – 9:00 PM – 10:00 PM (ET)

ANURAG :

W – 9:30 PM – 10:30 PM (ET)

LECTURES :

9:00 AM – 11:15 AM (ET)

M, T, W, R

Zoom ID :

979-4693-0650

COURSE WEBPAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL : bit.ly/sahay165

NOTE : ALL IMAGES ARE FROM (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1-19 ARE UPLOADED.

2. WW 10 - IS DUE ~~MON (1st AUG)~~ TUE (2nd AUG) AT 11:00 PM ET
   WW 11 - IS DUE ~~MON (1st AUG)~~ WED (3rd AUG) AT 11:00 PM ET

3. HARD WEBWORK DEADLINE : FRIDAY, 5th AUG

4. EXTRA OFFICE HOURS (SEE PREV. PAGE.)

5. FINAL EXAM ON THURSDAY (SCHEDULE + SAMPLE) → TONIGHT.

6. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE !

$$P(D)\,y = 0$$

$$P(D) = P_1(D)\,P_2(D)\,\dots\,P_k(D)$$

$$P_j(D)\,y = 0$$

$$P(D) = (D - r_0)^m$$

Lemma 8.2.5 Consider the differential operator $(D-r)^m$, where m is a positive integer, and r is a real or complex number. For any $u \in C^m(I)$,

$$(D-r)^m(e^{rx}u) = e^{rx}D^m(u). \tag{8.2.4}$$

Theorem 8.2.6 The differential equation $(D-r)^m y = 0$, where m is a positive integer and r is a real or complex number, has the following m solutions that are linearly independent on any interval:

$$e^{rx}, xe^{rx}, x^2e^{rx}, \ldots, x^{m-1}e^{rx}.$$

$y = e^{\lambda x} u \Rightarrow \quad D^m u = 0 \iff u \in \text{SPAN}\{1, \ldots, x^{m-1}\}$

Example 8.2.10 Solve the initial-value problem

$$y'' + 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 4.$$

$$\Big(\underbrace{D^2 + 4D + 4}_{P(D)}\Big) y = D^2 y + 4Dy + 4y = 0$$

$$P(r) = r^2 + 4r + 4 = (r+2)^2$$

$$P(D) = (D+2)^2 = (D-r)^m$$

THM SAYS. SOLN. ARE

$$\{ e^{-2x}, \ x e^{-2x} \}$$

$$W[e^{-2x}, x e^{-2x}] = \begin{vmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & e^{-2x} - 2x e^{-2x} \end{vmatrix}$$

$$\begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x}-2xe^{-2x} \end{vmatrix} \xrightarrow{CA_{12}(-x)} \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & e^{-2x} \end{vmatrix} = e^{-4x} \neq 0$$

<u>CHECK THEY ARE SOLNS.</u>

$$P(D)y = 0 \iff y \in \mathrm{Span}\{e^{-2x}, xe^{-2x}\}$$

$$\Rightarrow y = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$y(0) = 1, \quad y'(0) = 4$$

$$y(0) = c_1 \qquad y'(x) = -2c_1 e^{-2x} + c_2 x e^{-2x} - 2c_2 x e^{-2x}$$

$$y'(x) = -2c_1 e^{-2x} + c_2 x e^{-2x} - 2c_2 x e^{-2x}$$

$$y'(0) = -2c_1 + c_2 = 4$$

$$y(0) = c_1 = 1$$

$$c_1 = 1, \quad c_2 = 6$$

$$y = e^{-2x} + 6x e^{-2x} = e^{-2x}[1 + 6x]$$

Example 8.2.13 Determine the general solution to

DEG 9 $\leftarrow D^3(D-2)^2(D^2+1)^2y = 0.$

$P(D)y = 0$

$$P(D) = \underbrace{D^3}_{P_1}\underbrace{(D-2)^2}_{P_2}\underbrace{(D^2+1)^2}_{P_3}$$

$D^3 y = 0 \iff y \in \mathrm{Span}\{1, x, x^2\}$

$a=0, m=3$

$(D-2)^2 y = 0 \iff y \in \mathrm{Span}\{e^{2x}, xe^{2x}\}$

$a=2, m=2$

CAN'T APPLY THM.

$$(D^2+1)^2 y = 0$$

$$(r^2+1)^2$$

$$r^2+1=0 \Rightarrow r^2=-1$$
$$\Rightarrow r = \pm\sqrt{-1} = \pm i$$

$$(D+i)^2 (D-i)^2 y$$

$$(r^2+1) = (r+i)(r-i)$$
$$= r^2 + \cancel{ir} - \cancel{ir} \underbrace{- i^2}_{-(-1)}$$
$$= r^2+1$$

$$(D-r)^m y$$

$$(D+i)^2 y = 0$$

$r=-i, \quad m=2 \longrightarrow \{ e^{-ix}, \; x e^{-ix} \}$

$e^{-ix} = \boxed{(\cos x - i \sin x)}$ u_1

$x e^{-ix} = \overbrace{x(\cos x - i \sin x)}^{x\, u_1}$

$$(D-i)^2 y = 0 \longrightarrow \{ e^{ix}, \; x e^{ix} \}$$

$r=i, \quad m=2$

$e^{ix} = \boxed{\cos x + i \sin x}$ u_2

$x e^{ix} = \boxed{x(\cos x + i \sin x)}$ $x\, u_2$

$$y_1 = \frac{u_1 + u_2}{2}$$

$$= \frac{(\cos x - i\sin x) + (\cos x + i\sin x)}{2}$$

$$= \cos x$$

$$y_2 = \frac{u_2 - u_1}{2i}$$

$$= \frac{(\cos x + i\sin x) - (\cos x - i\sin x)}{2i}$$

$$= \frac{2i\sin x}{2i} = \sin x$$

$$x y_1 = \frac{x u_1 + x u_2}{2} \quad , \quad x y_2 = \frac{x u_2 - x u_1}{2}$$

$$(D^2+1)^2 y = 0$$

$$y \in \mathrm{SPAN}(y_1, y_2, xy_1, xy_2) \qquad [\mathbb{R}\text{-SOLNS.}]$$

$$= \mathrm{SPAN}(\cos x, \sin x, x\cos x, x\sin x)$$

$$\text{GEN. SOLN} \in \mathrm{SPAN}(1, x, x^2, e^{2x}, xe^{2x}, \cos x, \sin x, x\cos x, x\sin x)$$

$$P(r) = \overbrace{(r - r_1)^{m_1}}^{P_1} \cdots \overbrace{(r - r_k)^{m_k}}^{P_k}$$

[FUNDAMENTAL THEOREM OF ALGEBRA]

$\downarrow$

A POLYNOMIAL OF DEGREE d HAS EXACTLY d ROOTS IN $\mathbb{C}$.

$\Rightarrow$ SUFFICES TO SOLVE

$$(D - r)^m y = 0$$

$$P(r) = a_0 (r - r_1)(r - r_2) \cdots (r - r_d)$$

Theorem 8.2.7 Consider the differential equation

$$P(D)y = 0. \tag{8.2.7}$$

Let $r_1, r_2, \ldots, r_k$ be the distinct roots of the auxiliary equation, so that

$$P(r) = (r - r_1)^{m_1}(r - r_2)^{m_2} \cdots (r - r_k)^{m_k},$$

where m_i denotes the multiplicity of the root $r = r_i$.

1. If r_i is *real*, then the functions $\boxed{e^{r_i x}, xe^{r_i x}, \ldots, x^{m_i - 1}e^{r_i x}}$ are linearly independent solutions to Equation (8.2.7) on any interval.

2. If r_j is *complex*, say $r_j = a + ib$ (a and b are real, with $b \neq 0$), then the functions

$$\boxed{\begin{gathered} e^{ax}\cos bx,\ xe^{ax}\cos bx,\ \ldots,\ x^{m_j-1}e^{ax}\cos bx \\ e^{ax}\sin bx,\ xe^{ax}\sin bx,\ \ldots,\ x^{m_j-1}e^{ax}\sin bx \end{gathered}}$$

   corresponding to the conjugate roots $r = a \pm ib$ are linearly independent solutions to Equation (8.2.7) on any interval.

3. The n real-valued solutions $y_1, y_2, \ldots, y_n$ to Equation (8.2.7) that are obtained by considering the distinct roots $r_1, r_2, \ldots, r_k$ are linearly independent on any interval. Consequently, the general solution to Equation (8.2.7) is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x).$$

$a+bi$

$a-bi$

§9.1 FIRST ORDER LINEAR SYSTEM / §9.2 VECTOR FORMULATION

$t \to$ INDEPENDENT.

$x_1, \ldots, x_n \to$ DEPENDENT

$\frac{1}{3}$ rd OF MATH 164.

$\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$

$\vec{x}(t) \in \mathbb{R}^n$

(VECTOR FUNCTION)

$\vec{x} : I/\mathbb{R} \longrightarrow \mathbb{R}^n$

$x_j : I/\mathbb{R} \longrightarrow \mathbb{R}$

$\vec{x} = (x_1, \ldots, x_n)$

$$x_1, \ldots, x_n, \quad \frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}, \frac{d^2 x_1}{dt^2}, \ldots, \frac{d^k x_n}{dt^k}$$

$$\vec{b}(t) = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

DEFINITION 9.1.1

A system of differential equations of the form

$$\frac{d\vec{x}}{dt} \rightarrow \quad \begin{aligned} \frac{dx_1}{dt} &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t), \\ \frac{dx_2}{dt} &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + b_2(t), \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + b_n(t), \end{aligned} \tag{9.1.1}$$

$$\underbrace{\frac{d\vec{x}}{dt} = A(t)\,\vec{x}(t) + \vec{b}(t)}$$

FIRST-ORDER
LINEAR
VECTOR
DIFF.
EQN.

where the $a_{ij}(t)$ and $b_i(t)$ are specified functions on an interval I, is called a **first-order linear system**. If $b_1 = b_2 = \cdots = b_n = 0$, then the system is called **homogeneous**. Otherwise, it is called **nonhomogeneous**.

(OF DIFF. EQNS.)

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

$$A(t)\,\vec{x}(t) = \begin{bmatrix} \cdots \\ \vdots \ \ddots \\ \cdots \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$\vec{x}(t) = (x_1, \ldots, x_n)$$

$$\frac{d\vec{x}}{dt} = \left(\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt} \right)$$

$(f_1, \ldots, f_n)$

$x_j(t) = f_j(t)$

DEFINITION 9.1.3

By a **solution** to the system (9.1.1) on an interval I we mean an ordered n-tuple of functions $x_1(t), x_2(t), \ldots, x_n(t)$, which, when substituted into both sides of the system, yield the same result for all t in I.

VECTOR FORM:

Find all column vector functions $\mathbf{x}(t) \in V_n(I)$ satisfying the vector differential equation

$$\boxed{\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t).}$$

Example 9.1.4 Verify that

$$x_1(t) = -2e^{5t} + 4e^{-t}, \qquad x_2(t) = e^{5t} + e^{-t} \tag{9.1.2}$$

$$\vec{x} = \begin{bmatrix} -2e^{5t} + 4e^{-t} \\ e^{5t} + e^{-t} \end{bmatrix}$$

is a solution to the linear system of differential equations

$$\boxed{\begin{aligned} x_1' &= x_1 - 8x_2, \rightarrow 1 \\ x_2' &= -x_1 + 3x_2, \rightarrow 2 \end{aligned}}$$

(9.1.3)

(9.1.4)

on $(-\infty, \infty)$.

$$\begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} = \frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & -8 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = \begin{bmatrix} 1 & -8 \\ -1 & 3 \end{bmatrix}$$

$$\vec{b} = 0$$ (HOMOGENEOUS)

$$x_1(t) = -2e^{5t} + 4e^{-t} \quad , \quad x_1'(t) = -10e^{5t} - 4e^{-t}$$

$$x_2(t) = e^{5t} + e^{-t} \quad , \quad x_2'(t) = 5e^{5t} - e^{-t}$$

$$RHS_1 = x_1 - 8x_2 = \left(-2e^{5t} + 4e^{-t}\right) - 8\left(e^{5t} + e^{-t}\right)$$

$$= -10e^{t} - 4e^{-t} = x_1(t) = LHS_1$$

$$RHS_2 = -x_1 + 3x_2 = -\left(-2e^{5t} + 4e^{-t}\right) + 3\left(e^{5t} + e^{-t}\right)$$

$$= 5e^{5t} - e^{-t} = x_2'(t) = LHS_2$$

$\Rightarrow$ THIS IS A SOLUTION.

BREAK TILL

10:00 AM

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

Example 9.1.5 Solve the system

$$x_1' = x_1 + 2x_2,$$
$$x_2' = 2x_1 - 2x_2.$$

$$D = \frac{d}{dt}$$

$$Dx_1 = x_1 + 2x_2$$

$$Dx_2 = 2x_1 - 2x_2$$

$$\Rightarrow (D-1)x_1 - 2x_2 = 0$$

$$-2x_1 + (D+2)x_2 = 0$$

$$(D-1)x_1 - 2x_2 = 0 \quad -\text{I}$$

$$-2x_1 + (D+2)x_2 = 0 \quad -\text{II}$$

MULTIPLY II WITH 2 & MULTIPLY I WITH $(D+2)$, THEN ADD

$$(D+2)(D-1)x_1 \cancel{-2(D+2)x_2} = 0$$

$$+2(-2x_1) + \cancel{2(D+2)x_2}$$

$$\Rightarrow [(D+2)(D-1) - 4]x_1 = 0$$

$$\underbrace{\left[(D+2)(D-1)-4\right]}_{P(D)} x_1 = 0$$

$$P(r) = (r+2)(r-1) - 4$$

$$= r^2 + r - 2 - 4 = r^2 + r - 6$$

$$= (r+3)(r-2)$$

$\Rightarrow$ SOLNS. ARE IN SPAN $\{e^{-3t}, e^{2t}\}$

$$x_1 = c_1 e^{-3t} + c_2 e^{2t}$$

PLUG INTO (I)

$$(D-1)x_1 - 2x_2 = 0$$

$$(D-1)\left[c_1 e^{-3t} + c_2 e^{2t}\right] = 2x_2$$

$$x_2 = \frac{1}{2}\left[(D-1)\, c_1 e^{-3t} + (D-1) c_2 e^{2t}\right]$$

$$= \frac{1}{2}\left[-3c_1 e^{-3t} - c_1 e^{-3t} + (2c_2 e^{2t}) - c_2 e^{2t}\right]$$

$$x_2 = -2c_1 e^{-3t} + \frac{1}{2} c_2 e^{2t}$$

$$x_1 = c_1 e^{-3t} + c_2 e^{2t}$$

$$x_2 = -2c_1 e^{-3t} + \frac{1}{2} c_2 e^{2t}$$

$$\vec{x} = \begin{bmatrix} c_1 e^{-3t} + c_2 e^{2t} \\ -2c_1 e^{-3t} + \frac{1}{2} c_2 e^{2t} \end{bmatrix} = c_1 e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

DEFINITION 9.1.6

Solving the system (9.1.1) subject to n auxiliary conditions imposed at the *same* value of the independent variable is called an **initial-value problem**. Thus, the general form of the auxiliary conditions for an initial-value problem is:

$$x_1(t_0) = \alpha_1, \quad x_2(t_0) = \alpha_2, \quad \ldots, \quad x_n(t_0) = \alpha_n,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are constants.

$$\boxed{\begin{aligned} \frac{d\vec{x}}{dt} &= A(t)\,\vec{x}(t) + b(t) \\ \vec{x}(t_0) &= \vec{\alpha} \end{aligned}} \rightarrow \text{IVP IN VECTOR FORMULATION.}$$

Example 9.1.7 Solve the initial-value problem

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x_1' = x_1 + 2x_2, \qquad x_2' = 2x_1 - 2x_2,$$
$$x_1(0) = 1, \qquad x_2(0) = 0.$$

$$x_1 = c_1 e^{-3t} + c_2 e^{2t}$$
$$x_2 = -2c_1 e^{-3t} + \frac{1}{2} c_2 e^{2t}$$

$$1 = x_1(0) = c_1 e^{-3\cdot 0} + c_2 e^{2\cdot 0} = c_1 + c_2$$
$$0 = x_2(0) = -2c_1^{-3\cdot 0} + \tfrac{1}{2} c_2 e^{2\cdot 0} = -2c_1 + c_2/2$$

$$c_1 + c_2 = 1 \Rightarrow 5c_2 = 1 \Rightarrow c_2 = 1/5$$

$$-2c_1 + c_2/2 = 0 \Rightarrow c_2 = 4c_1 \Rightarrow c_1 = 4/5$$

$$x_1 = \frac{4}{5} e^{-3t} + \frac{1}{5} e^{-t}$$

$$x_2 = \frac{-8}{5} e^{-3t} + \frac{1}{10} e^{-t}$$

Q. WHAT ABOUT HIGHER ORDER VECTOR DIFFERENTIAL EQUATIONS /SYSTEMS OF LINEAR ODEs?

$$\boxed{\frac{d^2x}{dt^2}} - 4y = e^t$$

$$\frac{d^2y}{dt^2} + t^2\frac{dx}{dt} = \sin t,$$

2nd ORDER LINEAR SYSTEMS.

$\boxed{x_1 = x \quad , \quad x_2 = \frac{dx}{dt}}$, $\frac{dx_2}{dt} = \frac{d^2x}{dt^2}$

$\boxed{x_3 = y \quad , \quad x_4 = \frac{dy}{dt}}$, $\frac{dx_4}{dt} = \frac{d^2y}{dt}$

$$\frac{d^2x}{dt^2} - 4y = e^t \Leftrightarrow \frac{dx_2}{dt} - 4x_3 = e^t$$

$$\frac{d^2y}{dt^2} + t^2\frac{dx}{dt} = \sin t \Leftrightarrow \frac{dx_4}{dt} + t^2 x_3 = \sin t$$

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_3}{dt} = x_4$$

$$dx_1/dt = x_2$$

$$dx_2/dt = 4x_3 + e^t$$

$$dx_3/dt = x_4$$

$$dx_4/dt = -t^2 x_3 + \sin t$$

$$\frac{d\vec{x}}{dt} = \overbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -t^2 & 0 \end{bmatrix}}^{A(t)} \vec{x} + \underbrace{\begin{bmatrix} 0 \\ e^t \\ 0 \\ \sin t \end{bmatrix}}_{b(t)}$$

$$\frac{d\vec{x}}{dt} = A(t)\,\vec{x} + \vec{b}(t)$$

2nd LINEAR ODE (IN ONE DEPENDENT)

$$\boxed{\frac{d^2x}{dt^2}} + 4e^t\frac{dx}{dt} - 9t^2x = 7t^2.$$

$x_1 = x \qquad x_2 = \frac{dx}{dt} \quad \Rightarrow \quad \frac{dx_2}{dt} = \frac{d^2x}{dt^2}$

FOLSDE IN 2 VARIABLE

$$\frac{dx_2}{dt} + 4e^t x_2 - 9t^2 x_1 = 7t^2$$

$$\frac{dx_1}{dt} = x_2$$

$$V_n(I) = \{ \vec{x}(t) : \vec{x}(t) = (x_1(t), \dots, x_n(t)) \quad x_j : \mathbb{R} \longrightarrow \mathbb{R} \}$$

Theorem 9.2.1 The set $V_n(I)$ is a vector space.

$$x_j : I \longrightarrow \mathbb{R}$$

$$\vec{x}(t) = (x_1, \dots, x_n) \in \mathbb{R}^n$$

$$\vec{x} : I \longrightarrow \mathbb{R}^n$$

WRONSKIANS IN $V_n(I)$

DEFINITION 9.2.2

Let $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ be vectors in $V_n(I)$. Then the **Wronskian** of these vector functions, denoted $W[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n](t)$, is defined by

$$W[\underbrace{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n}_{\text{DEPENDS ON ORDER}}](t) = \det([\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)]).$$

Example 9.2.3 Determine the Wronskian of the column vector functions

$$\mathbf{x}_1(t) = \begin{bmatrix} e^t \\ 2e^t \end{bmatrix}, \qquad \mathbf{x}_2(t) = \begin{bmatrix} 3\sin t \\ \cos t \end{bmatrix}.$$

$n = 2$ (FUNCTIONS ARE $I \to \mathbb{R}^2$, TWO FUNCTIONS)

$$W[\vec{x}_1(t), \vec{x}_2(t)] = \det \begin{vmatrix} e^t & 3\sin t \\ 2e^t & \cos t \end{vmatrix}$$

$$\underbrace{}_{\vec{x}_1} \qquad \underbrace{}_{x_2}$$

$$= e^t \cos t - 6e^t \sin t$$

Theorem 9.2.4 Let $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ be vectors in $V_n(I)$. If $W[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n](t_0)$ is *nonzero* at some point t_0 in I, then $\{\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)\}$ is linearly independent on I.

$$c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \cdots + c_n \vec{x}_n(t) = \vec{0}$$

$0 \in V_n(I)$

$$= \underbrace{\begin{bmatrix} \vec{x}_1(t) & \cdots & x_n(t) \end{bmatrix}}_{X(t)} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$$

$\det X \neq 0$ at $t = t_0$

$\Rightarrow X(t_0)$ IS INVERTIBLE

$$X(t)\vec{c} = \vec{0}$$

$$\Rightarrow \vec{c} = X(t_0)^{-1} \cdot \vec{0} = \vec{0}$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

$$W[\vec{x}_1, \vec{x}_2](t) = e^t \ln t - 6e^t \sin t$$

$(t = 0)$

$$= e^0 \ln 0 - 6e^0 \sin 0$$

$$= 1 \neq 0$$

$\Rightarrow \vec{x}_1$ & $\vec{x}_2$ ARE L. I. AS ELEMENTS OF $V_2(I)$.

§9.3 GENERAL RESULTS FOR FOLDS.

Theorem 9.3.1 The initial-value problem

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0,$$

where $A(t)$ and $\mathbf{b}(t)$ are continuous on an interval I, has a unique solution on I.

Pf IS OMITTED.

$0 \in I.$

Theorem 9.3.2 The set of all solutions to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$, where $A(t)$ is an $n \times n$ matrix function that is continuous on an interval I, is a vector space of dimension n.

$$[D - A]\vec{x} = 0$$

PROOF: LET $\vec{x}_j$ BE THE UNIQUE SOLN

TO

$$\frac{d\vec{x}}{dt} = A(t)\,\vec{x}(t), \quad \vec{x}(0) = \vec{e}_j$$

$$\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow \text{POSITION } j.$$

CLAIM: $\vec{x}_1, \dots, \vec{x}_n$ ARE L.I.

$$W[\vec{x}_1, \dots, \vec{x}_n](0) = \det \begin{bmatrix} \vec{x}_1(0) & \vec{x}_2(0) & \dots & \vec{x}_n(0) \end{bmatrix}$$

$$= \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_n \end{bmatrix}$$

$$= \begin{vmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{vmatrix} = 1 \neq 0$$

CLAIM: $\{x_1, \dots, x_n\}$ SPANS THE SOLN SET.

Pf. $\dfrac{d\vec{x}}{dt} = A(t)\vec{x}(t)$

$\vec{x}(t) = (x_1(t), \dots, x_n(t))$

$c_j = x_j(0)$

$\vec{x}(0) = (c_1, \dots, c_n)$

$= c_1(1, 0 \dots, 0) + c_2(0, 1, \dots 0) + \dots = c_1\vec{e}_1 + c_2\vec{e}_2 + \dots + c_n\vec{e}_n$

$$\vec{y} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \cdots + c_n \vec{x}_n$$

$$\vec{y}(0) = c_1 \vec{e}_1 + \cdots + c_n \vec{e}_n = \vec{x}(0)$$

$$(\because \vec{x}_j(0) = \vec{e}_j$$

<u>UNIQUENESS</u> $\Rightarrow \vec{x} = \vec{y}$

$$= c_1 \vec{x}_1 + \cdots + c_n \vec{x}_n$$

$$\in \text{Span}\{\vec{x}_1, \ldots, \vec{x}_n\}$$

$$\boxed{\text{DIM SOL'N SET} = n}$$

DEFINITION 9.3.3

Let $A(t)$ be an $n \times n$ matrix function that is continuous on an interval I. Any set of n solutions, $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, to $\mathbf{x}' = A\mathbf{x}$ that is linearly independent on I is called a **fundamental solution set** on I. The corresponding matrix $X(t)$ defined by

$$X(t) = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$$

is called a **fundamental matrix** for the vector differential equation $\mathbf{x}' = A\mathbf{x}$.

<u>GENERAL SOLN.</u> : $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \ldots + c_n \vec{x}_n$

$\vec{x}(t) = c_1 \vec{x}_1(t) + \ldots + c_n \vec{x}_n(t)$

Example 9.3.5 Consider the vector differential equation

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where} \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix},$$

$n=2$

and let

$$\mathbf{x}_1(t) = \begin{bmatrix} -e^t \cos 2t \\ e^t \sin 2t \end{bmatrix}, \qquad \mathbf{x}_2(t) = \begin{bmatrix} e^t \sin 2t \\ e^t \cos 2t \end{bmatrix}.$$

$x_1(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\quad x_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(a) Verify that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a fundamental set of solutions for the vector differential equation on any interval, and write the general solution to the vector differential equation.

$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$

(b) Solve the initial-value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(PUTTING $t=0$)

and write the corresponding scalar solutions.

$\begin{bmatrix} -c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow$ $c_1 = -3$, $c_2 = 2$

$\vec{x} = -3\vec{x}_1 + 2\vec{x}_2 = \begin{bmatrix} 3e^t \cos 2t + 2e^t \sin 2t \\ -3e^t \sin 2t + 2e^t \cos 2t \end{bmatrix}$

NEED TO SHOW : ① $\vec{x}_1$ & $\vec{x}_2$ ARE SOLUTIONS

② $\vec{x}_1$ & $\vec{x}_2$ ARE L.I.

$\boxed{\text{① FOR } \vec{x}_1}$

$$\vec{x}_1 = \begin{bmatrix} -e^t \cos 2t \\ e^t \sin 2t \end{bmatrix}$$

$$\frac{d\vec{x}_1}{dt} = \begin{bmatrix} -e^t \cos 2t + 2e^t \sin 2t \\ e^t \sin 2t + 2e^t \cos 2t \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -e^t \cos 2t \\ e^t \sin 2t \end{bmatrix} = A\vec{x}_1$$

$\Rightarrow \vec{x}_1$ IS A SOLUTION.

(2) $\vec{x}_1$ & $\vec{x}_2$ ARE L. I.

$$W[\vec{x}_1, \vec{x}_2] = \det\begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix}$$

$$= \begin{vmatrix} -e^t \cos 2t & e^t \sin 2t \\ e^t \sin 2t & e^t \cos 2t \end{vmatrix}$$

$$= \left(-e^{2t} \cos^2 2t\right) - \left(e^{2t} \sin^2 2t\right)$$

$$= -e^{2t}\left(\cos^2 2t + \sin^2 2t\right) = -e^{2t} \neq 0$$