MATH 165 (SUMMER '22, SESH B2)

REMAINING
OFFICE HOURS :

ANURAG :
T – 10:00 PM – 11:00 PM (ET)

PABLO :
W – 9:30 PM – 10:30 PM (ET)

LECTURES :
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID :
979-4693-0650

COURSE WEBPAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL : bit.ly/sahay165

NOTE : ALL IMAGES ARE FROM (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1-20 ARE UPLOADED.

2. WW 10 - IS DUE WED (3rd AUG) AT 11:00 PM ET
   WW 11 - IS DUE FRI (5th AUG) AT 11:00 PM ET

3. HARD WEBWORK DEADLINE : FRIDAY, 5th AUG

4. WW 11 IS EXTRA CREDIT. A - SCORE WITHOUT WW11, S = MAX WITHOUT WW11
   B - SCORE IN WW 11

$$MAX\left(\frac{A+B}{S}, 1\right) * 25$$

5. EXTRA OFFICE HOURS (SEE PREV. PAGE.)

ANNOUNCEMENTS / NOTES

MAY CHANGE

6. FINAL EXAM ON THURSDAY (SCHEDULER + SAMPLE)

7. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE !

§9.4 VECTOR DIFF. EQUATIONS: NON-DEFECTIVE COEFF. MATRIX

$$\frac{d\vec{x}}{dt} = \underbrace{A(t)}_{n\times n \text{ MATRIX}}\vec{x}(t) + \underbrace{\vec{b}(t)}_{\vec{b}: I \to \mathbb{R}^n}$$

$\left(\begin{array}{ll} b \equiv 0 & \text{HOMOGENEOUS} \\ A(t) = A & \text{CONSTANT COEFFICIENT} \end{array} \right)$

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

($A \to n \times n$ CONSTANT MATRIX)

<u>GUESS</u>: $\vec{x}(t) = e^{\lambda t}\cdot\vec{v}$

$\vec{v} \to$ CONSTANT VECTOR

$$\frac{d\vec{x}}{dt} = \frac{d}{dt}\left(e^{\lambda t} \cdot \vec{v}\right) = \frac{d}{dt}\begin{bmatrix} e^{\lambda t} v_1 \\ e^{\lambda t} v_2 \\ \vdots \\ e^{\lambda t} v_n \end{bmatrix} = \begin{bmatrix} \lambda e^{\lambda t} v_1 \\ \vdots \\ \lambda e^{\lambda t} v_n \end{bmatrix}$$

$$= \left(\lambda e^{\lambda t}\right) \vec{v}$$

$$A\vec{x} = A\left(e^{\lambda t} \vec{v}\right) = e^{\lambda t} (A\vec{v})$$ "?

$$\left(\cancel{e^{\lambda t}}\right)(A\vec{v}) = \lambda \cancel{e^{\lambda t}} \vec{v}$$

$$\Rightarrow \quad A\vec{v} = \lambda \vec{v}$$

(EIGENVECTOR - EIGENVALUE EQUATION)

Theorem 9.4.1 Let A be an $n \times n$ matrix of real constants, and let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{v}$. Then

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

is a solution to the constant coefficient vector differential equation $\mathbf{x}' = A\mathbf{x}$ on any interval.

<u>Pf</u>: PLUG IT IN.
(SEE PREV. SLIDES)

Example 9.4.2 Find the general solution to

$$\begin{aligned} x_1' &= 2x_1 + x_2, \\ x_2' &= -3x_1 - 2x_2. \end{aligned} \tag{9.4.3}$$

$$\frac{d\vec{x}}{dt} = \underbrace{\begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}}_{A} \vec{x}$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - (1)(-3)$$

$$= \lambda^2 - 4 + 3 = \lambda^2 - 1$$

E.V. , $\lambda = 1$, $\lambda = -1$

$$(A - 1I)\vec{v} = 0$$

$$\begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \Rightarrow \quad v_1 + v_2 = 0$$

$\Rightarrow$ E.V. $= (-1, 1)$ [FOR $\lambda = 1$]

$$(A - (-1)I)\vec{v} = 0$$

$$\begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \Rightarrow \quad 3v_1 + v_2 = 0$$

E.V. $= (1, -3)$ [FOR $\lambda = -1$]

$$\vec{x} = e^{\lambda t} \cdot \vec{v}$$

① $\vec{x}_a = e^{t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $[\lambda = 1]$

INDEPENDENT?

② $\vec{x}_b = e^{-t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $[\lambda = -1]$

$$\vec{x} = c_1 \vec{x}_a + c_2 \vec{x}_b = \begin{bmatrix} -c_1 e^{t} + c_2 e^{-t} \\ c_1 e^{t} - 3c_2 e^{-t} \end{bmatrix}$$

$$\boxed{x_1(t) = -c_1 e^{t} + c_2 e^{-t}, \quad x_2(t) = c_1 e^{t} - 3c_2 e^{-t}}$$

$$W[\vec{x}_a, \vec{x}_b] = \begin{vmatrix} -e^{t} & e^{-t} \\ e^{t} & -3e^{-t} \end{vmatrix} = (3) - (1) = 2 \neq 0$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \vec{x}_a & x_b \end{array}$$

NON-DEFECTIVE.

Theorem 9.4.3 Let A be an $n \times n$ matrix of real constants. If A has n real linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, with corresponding real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (not necessarily distinct), then the vector functions $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ defined by

$$\mathbf{x}_k(t) = e^{\lambda_k t}\mathbf{v}_k, \qquad k = 1, 2, \dots, n,$$

for all t, are linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ on any interval. The general solution to this vector differential equation is

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n.$$

$$W[\vec{x}_1, \dots, \vec{x}_n] = \begin{vmatrix} e^{\lambda_1 t}\vec{v}_1 & e^{\lambda_2 t}\vec{v}_2 & \cdots & e^{\lambda_n t} v_n \end{vmatrix}$$

$$= e^{(\lambda_1 + \lambda_2 \cdots + \lambda_n)t} \begin{vmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{vmatrix}$$

$\neq 0$ $\qquad$ $\neq 0$ $\quad (\vec{v}_1, \dots, \vec{v}_n \text{ L.I.})$

Example 9.4.4 Find the general solution to $\mathbf{x}' = A\mathbf{x}$ if $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 4 & -3 \\ -2 & 2 & -1 \end{bmatrix}$.

E.Vs / E.V.s OF A.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 & -3 \\ -2 & 4-\lambda & -3 \\ -2 & 2 & -1-\lambda \end{vmatrix}$$

$$= (-\lambda)\begin{vmatrix} 4-\lambda & -3 \\ 2 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} -2 & -3 \\ -2 & -1-\lambda \end{vmatrix} + (-3)\begin{vmatrix} -2 & 4-\lambda \\ -2 & 2 \end{vmatrix}$$

$$= (-\lambda)\begin{vmatrix} 4-\lambda & -3 \\ 2 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} -2 & -3 \\ -2 & -1-\lambda \end{vmatrix} + (-3)\begin{vmatrix} -2 & 4-\lambda \\ -2 & 2 \end{vmatrix}$$

$$= -(\lambda+1)(\lambda-2)^2$$

$\lambda = -1$, $\lambda = 2$

$$(A - (-1)I)\vec{v} = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & 5 & -3 \\ -2 & 2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 9 & -9 \\ 0 & 6 & -6 \end{bmatrix}$$

IGNORE

$v_1 + 2v_2 - 3v_3 = 0$

$9v_2 - 9v_3 = 0$

$v_3 = \lambda$

$v_2 = \lambda$

$v_1 = \lambda$

(E.V.) $\vec{v} = (r, r, r)$

$(1, 1, 1) \to$ E.V.

$$(A - 2I)\vec{v} = \begin{bmatrix} -2 & 2 & -3 \\ -2 & 2 & -3 \\ -2 & 2 & -3 \end{bmatrix}$$

$-2v_1 + 2v_2 - 3v_3 = 0$

$v_2 = r$

$v_3 = s$

$v_1 = v_2 - \frac{3}{2} v_3$

$= r - 3/2\, s$

$$\vec{v} = (r - 3/2\, s,\ r,\ s) = r(1,1,0) + s(-3/2, 0, 1)$$

$$= r(1,1,0) + s/2\,(-3, 0, 2)$$

E. V. $(1, 1, 0)$

$(-3, 0, 2)$

$$\vec{x}_1 = e^{-\lambda t}(1, 1, 1)$$

$$\vec{x}_2 = e^{2t}(1, 1, 0)$$

$$x_3 = e^{2t}(-3, 0, 2)$$

$$\boxed{\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3}$$ → GENERAL SOLN.

BREAK TILL

1[illegible] :00 AM

DETOUR: COMPLEX EIGENVALUES

Example 7.1.9 Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 9 & 37 \\ -1 & -3 \end{bmatrix}$.

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 9-\lambda & 37 \\ -1 & -3-\lambda \end{vmatrix}$$

$$= (9-\lambda)(-3-\lambda) - (-1)(37)$$

$$= \lambda^2 - 6\lambda - 27 + 37$$

$$= \lambda^2 - 6\lambda + 10$$

$$\text{disc} = (-6)^2 - 4(10) = -4 < 0$$

$$\text{E.V.} = \frac{-(-6) \pm \sqrt{-4}}{2} = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm i$$

E.V. $= 3+i$

$(A - \lambda I)\vec{v} = 0$

$$\begin{bmatrix} 6-i & 37 \\ -1 & -6-i \end{bmatrix} = 0$$

$$\Big\downarrow P_{12}$$

$$\begin{bmatrix} -1 & -6-i \\ 6-i & 37 \end{bmatrix} \xrightarrow{M_1(-1)} \begin{bmatrix} 1 & 6+i \\ 6-i & 37 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6+i \\ 6-i & 37 \end{bmatrix} \xrightarrow{A_{12}(-6+i)} \begin{bmatrix} 1 & 6+i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$v_1 + (6+i)v_2 = 0$$

$$v_2 = t$$

$$v_1 = -(6+i)t$$

$$\vec{v} = (-(6+i)t,\ t) = t\,\underbrace{(-6-i,\ 1)}_{\text{E.V.}}$$

E.V. $= 3 - i$

$$\begin{bmatrix} 6+i & 37 \\ -1 & -6+i \end{bmatrix} = 0$$

$$\Big\downarrow P_{12}$$

$$\begin{bmatrix} -1 & -6+i \\ 6+i & 37 \end{bmatrix} \xrightarrow{M_1(-1)} \begin{bmatrix} 1 & 6-i \\ 6+i & 37 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6-i \\ 6+i & 37 \end{bmatrix} \xrightarrow{A_{12}(-6-i)} \begin{bmatrix} 1 & 6-i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$v_1 + (6-i)v_2 = 0$$

$$v_2 = t$$

$$v_1 = -(6-i)t$$

$$\vec{v} = (-(6-i)t, t) = t\underbrace{(-6+i, 1)}_{\text{E.V.}}$$

$$(3+i) \xrightarrow{E.V.} (-6-i, 1)$$

$$(3-i) \xrightarrow{E.V.} (-6+i, 1)$$

$$z = x+iy \longleftrightarrow \overline{z} = x-iy \quad \text{(COMPLEX CONJUGATE)}$$

$$(3-i) = \overline{3+i}$$

$$(-6+i, 1) = (\overline{-6-i}, \overline{1})$$

Theorem 7.1.8 Let A be an $n \times n$ matrix with *real* elements. If λ is a complex eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then $\overline{\lambda}$ is an eigenvalue of A with corresponding eigenvector $\overline{\mathbf{v}}$.

$$A\vec{v} = \lambda\vec{v} \xrightarrow[\text{CONJ.}]{\text{TAKE}} \overline{A}\,\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}$$

$$\overline{A} = A \quad (A \text{ IS REAL})$$

$$\Rightarrow \quad A\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}$$

Example 9.4.6 Find the general solution to the vector differential equation $\mathbf{x}' = A\mathbf{x}$ if

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

E.V.s & E.Vs

$$p(\lambda) = \det(A - \lambda I) = \det\begin{vmatrix} 0-\lambda & 2 \\ -2 & 0-\lambda \end{vmatrix} = \lambda^2 - (-2)(2)$$

$$= \lambda^2 + 4$$

$$\lambda^2 + 4 = 0 \Rightarrow \lambda^2 = -4 \Rightarrow \lambda = \pm\sqrt{-4} = \pm 2i$$

$$\lambda_1 = 2i$$

$$(A - \lambda_1 I)\vec{v} = 0 \rightarrow \begin{bmatrix} -2i & 2 \\ \boxed{-2 \quad -2i} & \end{bmatrix} \vec{v} = 0$$

$$-2i\, v_1 + 2 v_2 = 0$$

$$v_1 = t \Rightarrow v_2 = \frac{2i\, v_1}{2} = it$$

$$\therefore \vec{v} = (t, it) = t \boxed{(1, i)} \rightarrow \text{E.V.}$$

E.VALUE | E.VECTOR

① $2i \longleftrightarrow (1, i)$ $\qquad e^{\lambda t} \cdot \vec{v}$

② $-2i \longleftrightarrow (1, -i)$

$$\vec{x}_1(t) = e^{2it}(1, i) = (\cos 2t + i \sin 2t, -\sin 2t + i \cos 2t)$$

$$\vec{x}_2(t) = e^{-2it}(1, -i) = (\cos 2t - i \sin 2t, -\sin 2t - i \cos 2t)$$

$$e^{it} = \cos t + i \sin t \qquad \vec{x}_2 = \overline{\vec{x}_1}$$

Theorem 9.4.5 Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be real-valued vector functions. If

$$\mathbf{w}_1(t) = \mathbf{u}(t) + i\mathbf{v}(t) \qquad \text{and} \qquad \mathbf{w}_2(t) = \mathbf{u}(t) - i\mathbf{v}(t)$$

are complex conjugate solutions to $\mathbf{x}' = A\mathbf{x}$, then

$$u = \frac{w_1 + w_2}{2}$$

$$\mathbf{x}_1(t) = \mathbf{u}(t) \qquad \text{and} \qquad \mathbf{x}_2(t) = \mathbf{v}(t)$$

$$v = \frac{w_1 - w_2}{2i}$$

are themselves *real-valued* solutions of $\mathbf{x}' = A\mathbf{x}$.

$$\left.\begin{aligned} \vec{u}(t) &= (\cos 2t, -\sin 2t) \\ \vec{v}(t) &= (\sin 2t, \cos 2t) \end{aligned}\right\} \quad c_1\vec{u} + c_2\vec{v}$$

SK30

Example 9.4.7 Find the general solution to the vector differential equation $\mathbf{x}' = A\mathbf{x}$ if

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix}.$$

Theorem 9.4.8 Let A be an $n \times n$ matrix of real constants.

1. Suppose λ is a real eigenvalue of A with corresponding linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Then k linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x}_j(t) = e^{\lambda t}\mathbf{v}_j, \qquad j = 1, 2, \ldots, k.$$

2. Suppose $\lambda = a + ib$ is a complex eigenvalue of A with corresponding linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, where $\mathbf{v}_j = \mathbf{r}_j + i\mathbf{s}_j$. Then k complex-valued solutions to $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{u}_j(t) = e^{\lambda t}\mathbf{v}_j, \qquad j = 1, 2, \ldots, k$$

and $2k$ *real-valued* linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ are

$$\begin{aligned}
\mathbf{x}_{11}(t) &= e^{at}(\cos bt\ \mathbf{r}_1 - \sin bt\ \mathbf{s}_1), & \mathbf{x}_{12}(t) &= e^{at}(\sin bt\ \mathbf{r}_1 + \cos bt\ \mathbf{s}_1) \\
\mathbf{x}_{21}(t) &= e^{at}(\cos bt\ \mathbf{r}_2 - \sin bt\ \mathbf{s}_2), & \mathbf{x}_{22}(t) &= e^{at}(\sin bt\ \mathbf{r}_2 + \cos bt\ \mathbf{s}_2) \\
&\vdots & &\vdots \\
\mathbf{x}_{k1}(t) &= e^{at}(\cos bt\ \mathbf{r}_k - \sin bt\ \mathbf{s}_k), & \mathbf{x}_{k2}(t) &= e^{at}(\sin bt\ \mathbf{r}_k + \cos bt\ \mathbf{s}_k)
\end{aligned}$$

Further, the set of all solutions to $\mathbf{x}' = A\mathbf{x}$ obtained in this manner is linearly independent on any interval.

§8.3 THE METHOD OF UNDETERMINED COEFFICIENTS

$$P(D)\,y = F(x) \quad - \text{(I)}$$

y_p SOLVES (I)

y SOLVES (I)

$$P(D)(y - y_p) = P(D)y - P(D)y_p$$
$$= F(x) - F(x)$$
$$= 0$$

$\Rightarrow$ $y - y_p$ SOLVES $P(D)y = 0$

$$y - y_p = y_c$$

$$y = \underbrace{y_c}_{\text{GEN. SOLN. OF } P(D)y=0} + y_p$$

$\rightarrow$ PARTICULAR SOLN.

"ANNIHILATOR"

SUPPOSE $A(D)$ s.t.

$A(D) \to$ POLYNOMIAL DIFFERENTIAL OPERATOR

$$A(D)\,F(x) = 0$$

$A(D)$ ANNIHILATES $F(x)$

$\Updownarrow$

$F(x)$ IS A SOLUTION TO $A(D)\,y = 0$

Example 8.3.1 Determine the general solution to

$$\underbrace{(D+3)(D-3)}_{P(D)}y = \underbrace{10e^{2x}}_{F(x)}$$

$$A(D)\left[10e^{2x}\right] = 0$$

$$y = e^{\lambda x} \longleftrightarrow \text{solves } (D-\lambda)y = 0$$

$$A(D) = D - 2$$

$$(D-2)(10e^{2x}) = \left[\frac{d}{dx}(10e^{2x}) - 2(10e^{2x})\right]$$

$$= \left[20e^{2x} - 20e^{2x}\right] = 0$$

$$(D-3)(D+3)\,y = 10\,e^{2x}$$

APPLY $A(D) = D-2$ ON BOTH SIDES.

$$(D-2)(D-3)(D+3)\,y = (D-2)\left(10\,e^{2x}\right) = 0$$

$$\underbrace{(D-2)(D-3)(D+3)}_{Q(D)}\,y = 0$$ (HOMOGENEOUS)

$$Q(\lambda) = (\lambda-2)(\lambda-3)(\lambda+3)$$

$\rightarrow \lambda = 2$

$\rightarrow \lambda = 3$

$\rightarrow \lambda = -3$

$$y = \overbrace{\boxed{c_1 e^{2x}}}^{y_p} + \overbrace{\boxed{c_2 e^{3x} + c_3 e^{-3x}}}^{y_c}$$

$$\underbrace{(D-2)}_{A(D)}\underbrace{(D+3)(D-3)}_{P(D)}\,y = 0$$

c_1 IS NOT A PARAMETER.

c_2 & c_3 ARE PARAMETERS.

$y = c_1 e^{2x}$ $\longrightarrow$ PLUG INTO ORIGINAL EQUATION.

$$(D+3)(D-3)\left(c_1 e^{2x}\right) = 10 e^{2x}$$

$$\begin{aligned} LHS &= \left(D^2 - 3\right)\left(c_1 e^{2x}\right) \\ &= D^2 c_1 e^{2x} - 3 c_1 e^{2x} \\ &= 4 c_1 e^{2x} - 3 c_1 e^{2x} \\ &= c_1 e^{2x} \end{aligned}$$

$RHS = 10 e^{2x}$

$\Rightarrow \boxed{c_1 = 10}$

$y_p = 10 e^{2x}$

GEN SOLN. $Y = Y_P + Y_C$

$$\boxed{Y = 10e^{2x} + C_2 e^{3x} + C_3 e^{-3x}}$$

$y = e^{\lambda x} \leftrightarrow$ SOLVES $(D-\lambda)y=0$

Example 8.3.3 Determine the general solution to

$$(D-4)(D+1)y = \boxed{15e^{4x}}.$$

$$A(D) = D - 4$$

$$(D-4)(D-4)(D+1)\,y = (D-4)(15e^{4x})$$

$$= D(15e^{4x}) - 60e^{4x}$$

$$= 15\frac{de^{4x}}{dx} - 60e^{4x}$$

$$= 60e^{4x} - 60e^{4x} = 0$$

$$(D-4)^2(D+1)\,y = 0$$

$$Q(\lambda) = (\lambda-4)^2(\lambda+1)$$

$\lambda = 4$

$\lambda = -1$

$$(D-4)^2 (D+1)\, y = 0$$

$$Q(\lambda) = (\lambda-4)^2(\lambda+1) \quad \begin{matrix} \nearrow \lambda = 4 \\ \searrow \lambda = -1 \end{matrix}$$

$$(D-4)^2 (x e^{4x}) = 0$$

$$(D-\lambda)^m y = 0$$

$y \in$ SPAN $\{ e^{\lambda x}, x e^{\lambda x}, \dots, x^{m-1} e^{\lambda x} \}$

$$y = c_1 e^{4x} + c_2 x e^{4x} + c_3 e^{-x}$$

$$= \underbrace{\boxed{c_1 e^{4x} + c_3 e^{-x}}}_{y_c} + \underbrace{\boxed{c_2 x e^{4x}}}_{y_p}$$

$$(D-4) \quad (D-4)(D+1)\, y = 0$$

PLUG y_p BACK INTO THE ORIGINAL EQUATION :

$$(D-4)(D+1)\,y = 15e^{4x}$$

$$LHS = (D-4)(D+1)\left[c_2 x e^{4x}\right]$$

$$= \left(D^2 - 3D - 4\right)\left(c_2 x e^{4x}\right)$$

$$\frac{d}{dx}\left(xe^{4x}\right) = e^{4x} + 4xe^{4x}, \quad \frac{d^2}{dx^2}\left(xe^{4x}\right) = 8e^{4x} + 16xe^{4x}$$

$$(D^2 - 3D - 4)(c_2 x e^{4x})$$

$$c_2\left[8e^{4x} + \cancel{16x e^{4x}} - 3\left(e^{4x} + \cancel{4x e^{4x}}\right) - 4\left(\cancel{x e^{4x}}\right)\right]$$

$$= c_2\left[5e^{4x} + 0\right] = 5c_2 e^{4x} = LHS$$

$$RHS = 15e^{4x}$$

$$c_2 = 3$$

$$y_p = 3x e^{4x}$$

$$y_c = c_1 e^{4x} + c_3 e^{-x}$$

$$y = y_p + y_c$$

$$= 3x e^{4x} + c_1 e^{4x} + c_3 e^{-x}$$