MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:

M – 9:00 PM – 10:00 PM (ET)

F – 3:00 PM – 4:00 PM (ET)

LECTURES:

9:00 AM – 11:15 AM (ET)

M, T, W, R

Zoom ID:

979-4693-0650

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURE 2 ARE UPLOADED.

2. WW 00 - DUE YESTERDAY (28th JUNE) AT 11:00 PM ET
   WW 01 - DUE SATURDAY (2nd JULY) AT 11:00PM ET
   WW 02 - DUE TUESDAY (5th JULY) AT 11:00 PM ET

3. OFFICE HOURS UPDATED (SEE PREVIOUS PAGE)

4. EXAM TIMES TO BE REVISED.

5. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE !

§ 1.3 THE GEOMETRY OF FIRST ORDER DIFF. EQNS. (CONTD.)

1st ORDER ODE : $\dfrac{dy}{dx} = f(x,y)$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

NOTE 1 : $dy/dx \rightsquigarrow$ SLOPE OF THE TANGENT

NOTE 2 : GENERAL SOLN. IS A ONE-PARAMETER FAMILY OF CURVES.

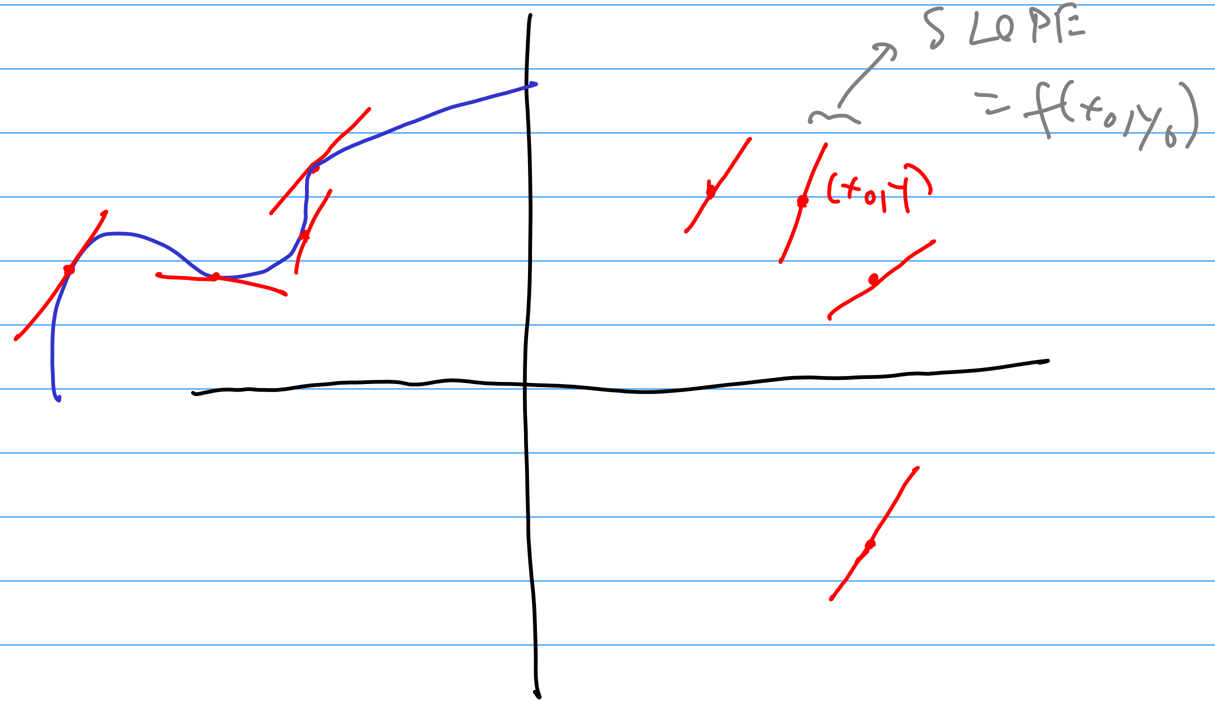
$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$$

EXISTENCE : A SOLUTION CURVE PASSES THROUGH (x_0, y_0)

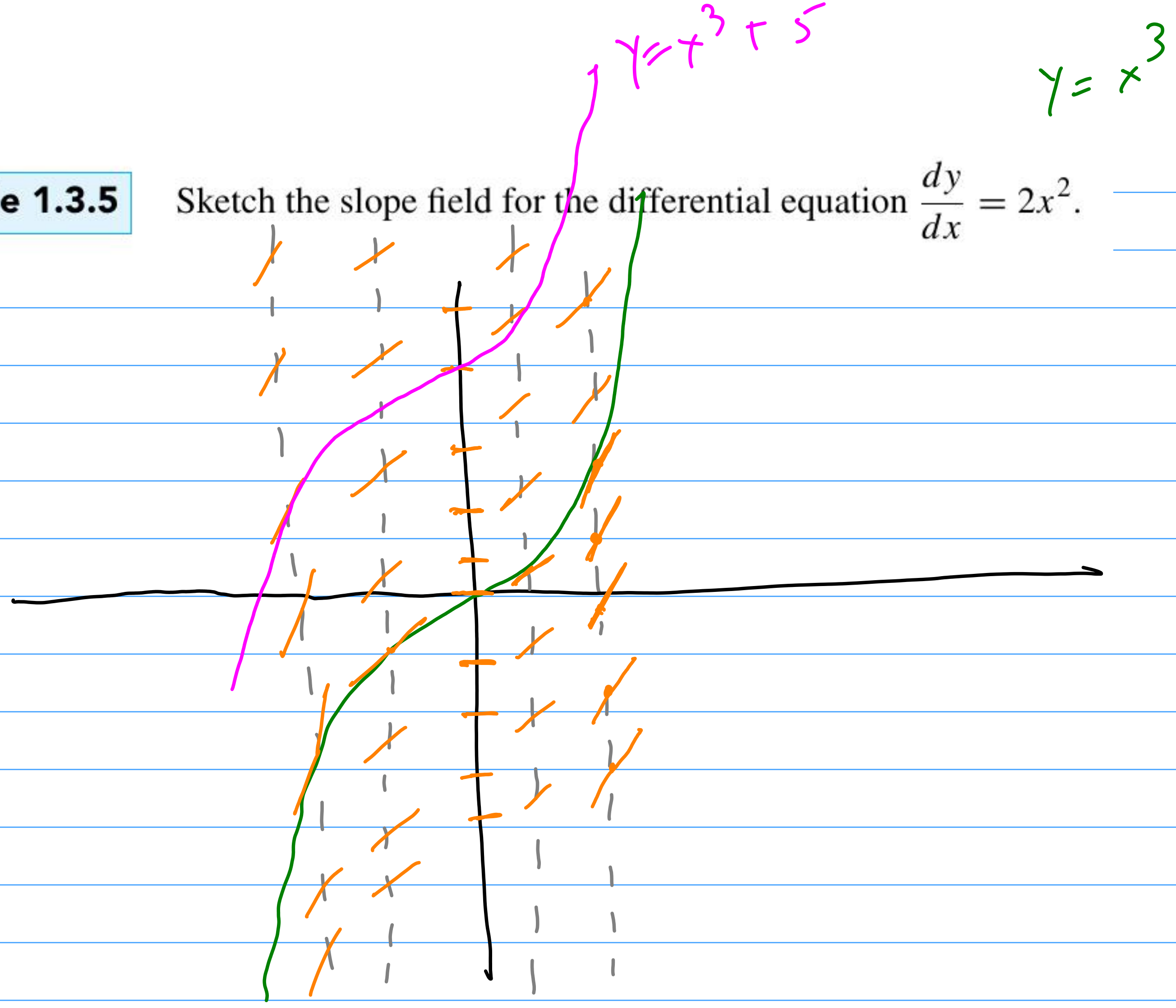
UNIQUENESS : SOLUTION CURVES DO NOT INTERSECT.

RECALL: SLOPE FIELDS

$$\frac{dy}{dx} = f(x,y)$$



Example 1.3.5 Sketch the slope field for the differential equation $\dfrac{dy}{dx} = 2x^2$.



$k \in \mathbb{R}$, FIXED

RECALL:

1. ISOCLINES $\quad f(x,y) = k \longrightarrow$ CURVE GIVEN BY THIS EQUATION

RECALL:

2. EQUILIBRIUM SOLUTION

(A SOLUTION OF THE FORM $y =$ CONSTANT)

$$f(x, \text{CONST}) = 0$$

$y =$ CONST SOLVES THE EQN. $\longleftarrow \quad \dfrac{dy}{dx} = \dfrac{d}{dx}(\text{CONST}) = 0 = f(x, \text{CONST})$

3. CONCAVITY $\dfrac{d^2y}{dx^2} \longleftarrow \left(\text{KNOW: } \dfrac{dy}{dx} = f(x,y)\right)$

IMPLICIT DIFF.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(f(x,y)\right)$$

$$\left.\frac{d^2y}{dx^2}\right|_{(x_0,y_0)} > 0 \longrightarrow \text{CONCAVE UP}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(x_0,y_0)} < 0 \longrightarrow \text{CONCAVE DOWN}$$

Example 1.3.6 Sketch the slope field and some approximate solution curves for the differential equation

$$\frac{dy}{dx} = y(2-y). = f(x,y) \tag{1.3.4}$$

ISOCLINES

$$f(x,y) = k$$

$$\Updownarrow$$

$$y(2-y) = k$$

$$\Rightarrow \quad y^2 - 2y + k = 0$$

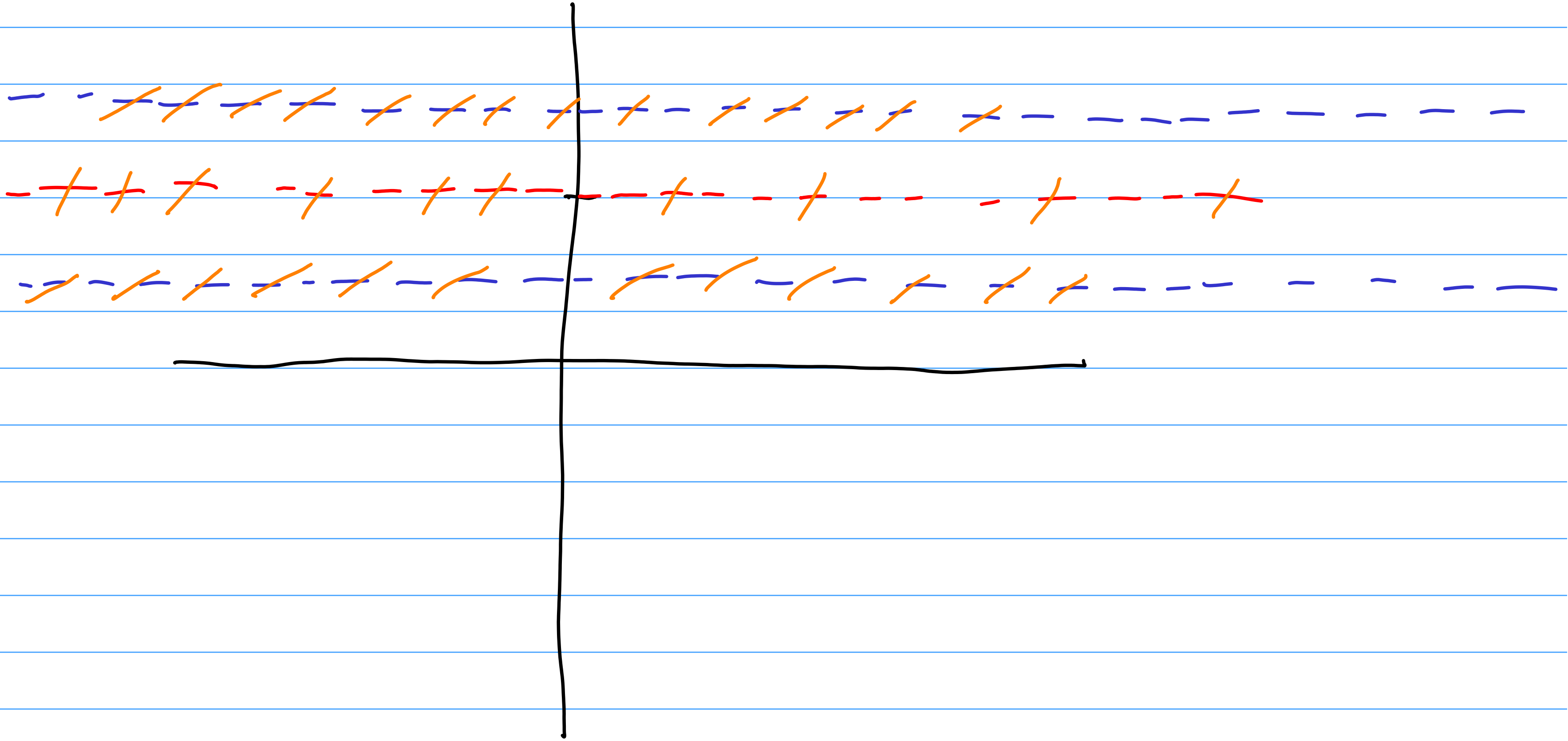
$$\Rightarrow y = \frac{2 \pm \sqrt{4 - 4k^2}}{2} = 1 \pm \sqrt{1 - k^2}$$

$$f(x,y) = y(2-y)$$

EQ. SOLNs: $y = 0$ & $y = 2$

$f(t,y) = 1/2$

$f(x,y) = 1$



EQ SOLTNS

$$\frac{dy}{dx} = y(2-y) \quad - \text{I}$$

CONCAVITY :

$$\frac{d}{dx}(\text{I}) : \frac{d^2y}{dx^2} = \frac{d}{dx}\left(y(2-y)\right)$$

$$= \frac{dy}{dx}(2-y) + y\frac{d}{dx}(2-y)$$

$$= (2-y)\frac{dy}{dx} - y\frac{dy}{dx}$$

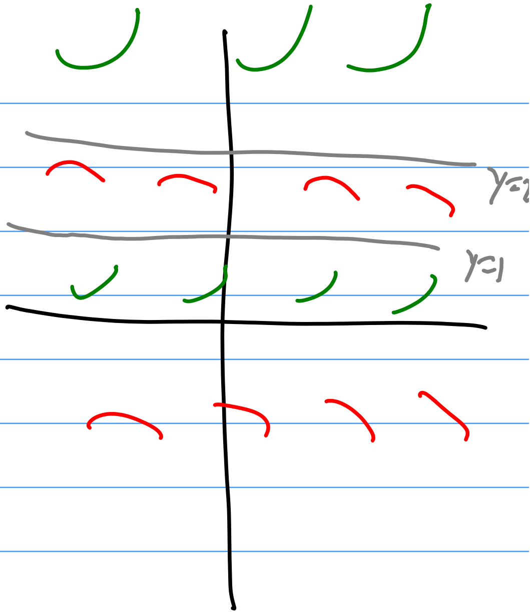
$$= (2-2y)\frac{dy}{dx}$$

$\text{I} \Rightarrow$

$$\frac{d^2y}{dx^2} = (2-2y)(y)(2-y)$$

$$= 2y(1-y)(2-y)$$

y	$y<0$	$0<y<1$	$1<y<2$	$y>2$
y	$-\ -\ -$	$+\ +$	$+\ +$	$+\ +$
$1-y$	$+\ +\ +$	$+\ +$	$-\ -$	$-\ -$
$2-y$	$+\ +\ +$	$+\ +$	$+\ +$	$-\ -$
$\frac{d^2y}{dt^2}=2y(1-y)(2-y)$	$-$	$+$	$-$	$+$



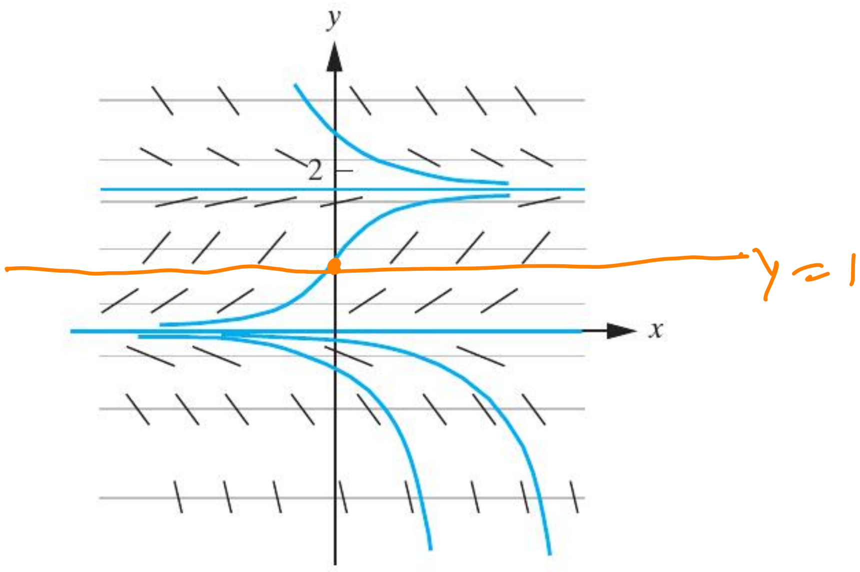


Figure 1.3.6: Hand-drawn slope field, isoclines, and some solution curves for the differential equation $\dfrac{dy}{dx} = y(2 - y)$.

y = 2, y = 0

EQ. SOLNS.

$$y(x) = \frac{2ce^{2x}}{ce^{2x} - 1}$$

Example 1.3.7 Sketch the slope field for the differential equation

$$\frac{dy}{dx} = y - x.$$

$\frac{d}{dx}(x-1) = 1$

$(x+1) - x = 1$

ISOCLINES: $f(x,y) = y - x = k$

$\boxed{y = x + k} \longleftrightarrow f(x,y) = k$

EQUILIBRIUM SOLNS: $f(x,y) = \boxed{y - x}$

HOME!

$c - x \neq 0$
$\forall c$

$y = x$ IS $\boxed{NO?}$ AN EQ. SOLN!

$$\frac{dy}{dx} = y - x \qquad - \text{(I)}$$

$$\frac{d}{dx}(\text{I}) : \quad \frac{d^2y}{dx^2} = \frac{d}{dx}(y-x)$$

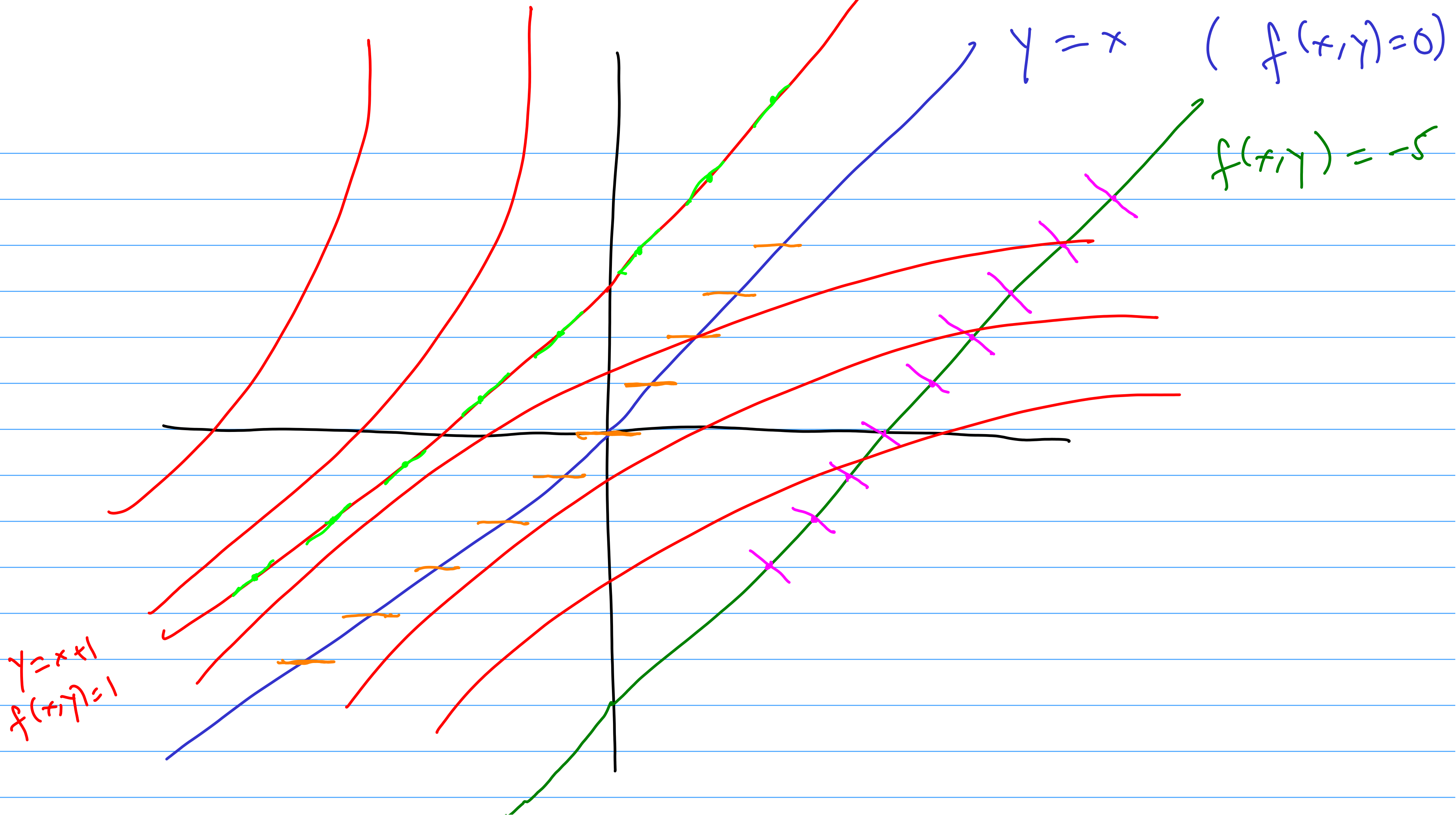
$$= \frac{dy}{dx} - 1$$

$$\frac{d^2y}{dx^2} = y - x - 1 \qquad (\text{By (I)})$$

$y - x - 1 > 0$ CONCAVE UP

$y - x - 1 < 0$ CONCAVE DOWN

y	vs	$x+1$	
y	$>$	$x+1$	UP
y	$=$	$x+1$	—
y	$<$	$x+1$	DOWN



y = x (f(x,y) = 0)
f(x,y) = -5
y = x + 1
f(x,y) = 1

$\boxed{\text{§ 1.4 SEPARABLE DIFF. EQNS.}}$

DEFINITION 1.4.1

A first-order differential equation is called **separable** if it can be written in the form

$$p(y)\frac{dy}{dx} = q(x). \tag{1.4.1}$$

$$\frac{dy}{dx} = f(x,y)$$

$$\rightarrow q(x)/p(y)$$

$$f(x,y) = x^2 y$$

$$\frac{dy}{dx} = x^2 y \Rightarrow \underbrace{\frac{1}{y}\frac{dy}{dx}}_{\text{NO } x} = \underbrace{x^2}_{\text{NO } y}$$

$$\frac{dy}{dx} = y - x \longrightarrow \text{NOT SEPARABLE}$$

$$\underset{\text{NO } x}{(\quad)}\frac{dy}{dx} = \underset{\text{NO } y}{(\qquad)}$$

$$p(y)\frac{dy}{dx} = q(x) \longrightarrow (1.4.1)$$

Theorem 1.4.2 If $p(y)$ and $q(x)$ are continuous, then Equation (1.4.1) has the general solution

$$\int p(y)\,dy = \int q(x)\,dx + c \tag{1.4.2}$$

where c is an arbitrary constant.

$Q \rightarrow$ ANTIDER. OF q

$P \rightarrow$ " OF p

$$p(y)\frac{dy}{dx} = q(x)$$

$$\frac{d}{dx}Q(x) = q(x), \qquad \frac{d}{dy}P(y) = p(y)$$

$$\frac{d}{dx}\left[P(y)\right] \underset{\text{(CHAIN RULE)}}{=} \left(\frac{dy}{dx}\right)\left(\frac{d}{dy}P(y)\right) = p(y)\frac{dy}{dx} = q(x)$$

$$\frac{d}{dx}\left[P(y)\right] = q(x)$$

$$P(y) = \int q(x)\,dx + C$$

$$\boxed{\int p(y)\,dy = \int q(x)\,dx + C}$$

DON'T MEMORISE

$$\frac{dy}{dx} = \frac{x}{3y^2} \longrightarrow 3y^2 \frac{dy}{dx} = x$$

$$\int 3y^2 \, dy = \int x \, dx + C$$

$$y^3 = \frac{x^2}{2} + C$$

$$\Rightarrow y = \left(\frac{x^2}{2} + C\right)^{1/3}$$

$C \longrightarrow$ CONSTANT OF INT.

/ PARAMETER

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

Example 1.4.3 Solve $(e^y + 4y^2)\dfrac{dy}{dx} = 9xe^{3x}$.

$$\boxed{\int (e^y + 4y^2)\,dy} = \boxed{\int (9x\,e^{3x})\,dx} + C$$

$$\int u\,dv = uv - \int v\,du$$

$$\int (e^y + 4y^2)\,dy = \int e^y\,dy + \int (4y^2)\,dy$$

$$= e^y + \frac{4}{3}y^3$$

$u = 9x \qquad dv = e^{3x}dx$

$$\int 9x\,e^{3x}\,dx = \overset{u}{(9x)}\overset{v}{\left(\frac{e^{3x}}{3}\right)} - \int \overset{v}{\left(\frac{e^{3x}}{3}\right)} \underbrace{(9\,dx)}_{du}$$

$$3x e^{3x} - \int 3 e^{3x}\, dx$$

$$= 3x e^{3x} - e^{3x}$$

$$e^{y} + \frac{4}{3} y^{3} = 3x e^{3x} - e^{3x} + C$$

Example 1.4.4 Find all solutions to

$$y' = -2xy^2.$$

$y \neq 0$

N.B. $y=0$ IS A SOLN.

$$\frac{1}{y^2}\,dy = (-2x)\,dx$$

$$\boxed{\int \frac{dy}{y^2}} = \boxed{\int (-2x)\,dx} + C$$

$$\int \frac{dy}{y^2} = -\frac{1}{y}, \quad \int(-2x)\,dx = -x^2$$

$$-\frac{1}{y} = -x^2 + C$$

$$\Rightarrow \boxed{y = \frac{1}{x^2 - C}}$$

$$y' = \frac{-2x}{(x^2-c)^2}$$

(CHAIN RULE)

$$x \mapsto x^2 - c \mapsto \frac{1}{x^2-c}$$

OR

(QUOTIENT RULE)

$$-2xy^2 = \frac{-2x}{(x^2-c)^2}$$

SOLN SET:

$$y = 0$$

$$\& \quad y = \frac{1}{x^2 - c} \quad \forall c \in \mathbb{R}$$

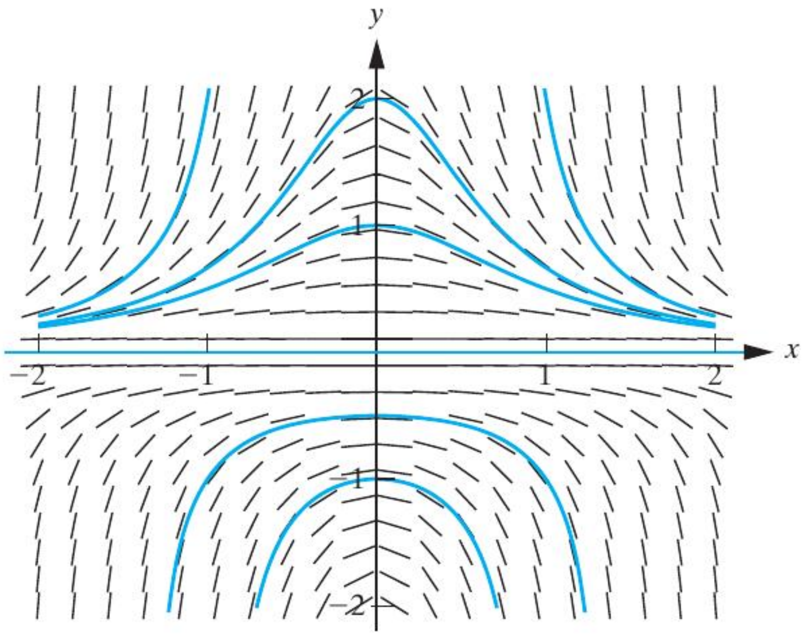


Figure 1.4.1: The slope field and some solution curves for the differential equation $\dfrac{dy}{dx} = -2xy^2$.

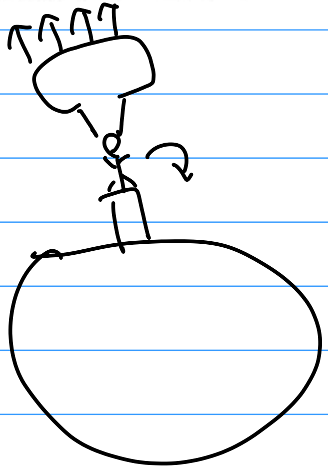
BREAK TILL

10:10 AM ET

$v(0) = 0$

Example 1.4.5 An object of $\boxed{\text{mass } m}$ falls from rest, starting at a point near the earth's surface. Assuming that the air resistance is proportional to the velocity of the object, determine the subsequent motion.

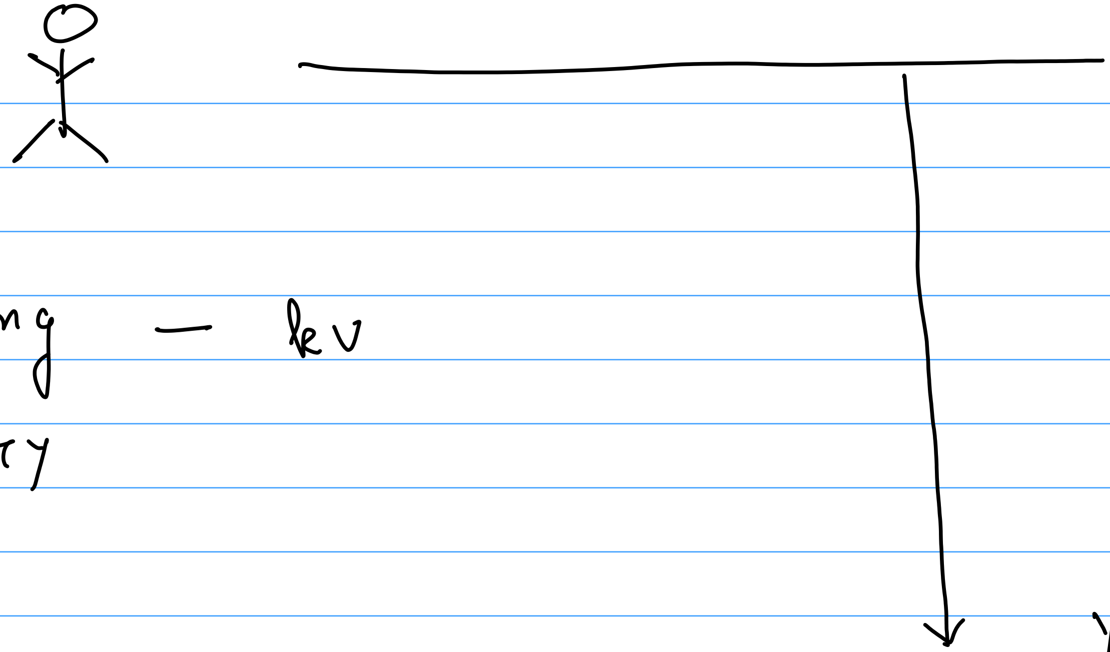
PARACHUTES



$ma = F$

FORCE DUE TO GRAVITY $= mg$

FORCE DUE TO AIR RESISTANCE $\propto v \Rightarrow -kv \quad (k > 0)$



$y = 0$

y +ve

$$ma = mg - kv$$

$v \rightarrow$ VELOCITY

$$a = \frac{dv}{dt}$$

$$m\frac{dv}{dt} = mg - kv$$

$\boxed{m, g, k} \rightarrow$ CONSTANT

$$\frac{m\,dv}{mg - kv} = dt$$

$$\Rightarrow \quad \frac{dv}{1 - \left(\frac{k}{mg}\right)v} = g\,dt$$

$$\int \frac{dv}{1 - \left(\frac{k}{mg}\right)v} = \underbrace{\int g\,dt}_{gt} + C$$

$$\int \frac{dv}{1-\left(\frac{k}{mg}\right)v}$$

$$u = 1 - \left(\frac{k}{mg}\right)v$$

$$du = \frac{-k}{mg}\,dv$$

$$\left(dv = \frac{-mg}{k}\,du\right.$$

$$\int \frac{\left(-mg/k\right)du}{u} = \frac{-mg}{k}\int \frac{du}{u}$$

$$= \left(-mg/k\right)\ln u$$

$$-\frac{mg}{k} \ln\left(1 - \frac{k}{mg} v\right) = gt + C \quad \text{- (I)}$$

$$t = 0 \Rightarrow v(0) = 0$$

$$\therefore \cancel{-\frac{mg}{k} \ln\left(1 - 0\right)} = \cancel{g \times 0} + C$$

$$C = 0$$

$$\text{(I)}: \quad -\frac{m\cancel{g}}{k} \ln\left(1 - \frac{k}{mg} v\right) = \cancel{g}t$$

$$\ln\left(1 - \frac{k}{mg} v\right) = -\frac{k}{m} t$$

$$1 - \frac{k}{mg} v = e^{-\frac{k}{m} t}$$

$$\Rightarrow \quad v = \frac{mg}{k} - \frac{mg}{k} e^{-\frac{k}{m} t}$$

$$v(t) = \frac{mg}{k}\left[1 - \underbrace{e^{-kt/m}}_{(t \to \infty)}\right]$$

$$k/m > 0$$

$v(t) \longrightarrow \dfrac{mg}{k}$ AS $t \longrightarrow \infty$

TERMINAL VELOCITY

Example 1.4.6 A hot metal bar whose temperature is 350°F is placed in a room whose temperature is constant at $\boxed{70°F}$. After $\boxed{\text{two minutes, the temperature of the bar is 210°F.}}$ Using Newton's law of cooling, determine

T_m

1. the temperature of the bar after four minutes. (ANS: 140°F)
2. the time required for the bar to cool to 100°F.

$$\frac{dT}{dt} \propto (T - T_m)$$

SURROUNDING TEMP $\leftrightarrow$ ROOM TEMP.

$$\frac{dT}{dt} = -k(T - T_m)$$

$k > 0$

$$\frac{dT}{dt} = -k(T - 70)$$

???

$T(0) = 350$

$T(2) = 210$

Q1. $T(4) = ??? \longrightarrow 140$

Q2. FOR WHAT t IS $T(t) = 100$

$$\frac{dT}{dt} = -k(T-70)$$

$$\Rightarrow \int_{\boxed{350}}^{\boxed{210}} \frac{dT}{T-70} = \int_{\boxed{0}}^{\boxed{2}} -k\,dt$$

$$\int_{350}^{210} \frac{dT}{T-70} = \int_0^2 -k\,dt = -kt\Big]_0^2 = -2k - 0$$

$$= -2k$$

$$\parallel$$

$$\ln(T-70)\Big]_{350}^{210} = \ln(210-70) - \ln(350-70)$$

$$= \ln(140) - \ln(280)$$

$$= \ln\left(\frac{140}{280}\right) = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$-2k = -\ln 2 \Rightarrow k = \frac{\ln 2}{2}$$

$$\frac{dT}{dt} = -\frac{\ln 2}{2}\left(T - 70\right)$$

$$\int_{350}^{T(t)} \frac{dT}{T-70} = \int_0^t -\frac{\ln 2}{2}\, dt = \left(\frac{-\ln 2}{2}\right) t$$

$$= \ln\left(T-70\right)\Big]_{350}^{T(t)} = \ln\left(T - 70\right) - \ln(350-70)$$

$$= \ln(T-70) - \ln(280)$$

$$= \ln\left(\frac{T-70}{280}\right)$$

$$e^{ab} = (e^a)^b$$

$$\ln\left(\frac{T-70}{280}\right) = \left(\frac{-\ln 2}{2}\right) t$$

$$\frac{T-70}{280} = \exp\left(\frac{-\ln 2}{2} t\right)$$

$$= e^{\frac{-\ln 2}{2} t} = \left(e^{\ln 2}\right)^{-t/2}$$

$$= 2^{-t/2}$$

$$\frac{T-70}{280} = 2^{-t/2} \Rightarrow \boxed{T(t) = 70 + (280) \cdot 2^{-t/2}}$$

$$T(4) = 70 + 280 \cdot \left(2^{-4/2}\right)$$

$$= 70 + 280 \cdot \left(2^{-2}\right)$$

$$= 70 + 70 = 140$$

$$T(t) = 100$$

$$\Rightarrow 70 + 280 \cdot 2^{-t/2} = 100$$

$$\Rightarrow 280 \cdot 2^{-t/2} = 100 - 70 = 30$$

$$2^{-t/2} = \frac{30}{280} = \frac{3}{28} \Rightarrow \frac{-t}{2} = \log_2\left(\frac{3}{28}\right)$$

$$t = -2\log_2\left(\frac{3}{28}\right) = 2\log_2\left(\frac{28}{3}\right)$$

SKIP

Example 1.4.8 In a certain chemical reaction 5 g of chemical C are formed when 2 g of chemical A react with 3 g of chemical B. Initially there are 10 g of A and 24 g of B present, and after 5 min, 10 g of C has been produced. Determine the amount of C that is produced in 15 min.

$$(y \neq 0)$$

$$\frac{dy}{dx} = -2xy$$

$$\underbrace{\int \frac{dy}{y}}_{\ln y} = \underbrace{\int -2x\,dx}_{-x^2} + C$$

$$\Rightarrow \quad \ln y = -x^2 + C \quad \Rightarrow \quad y = e^{\ln y} = e^{-x^2+C}$$

$$y = e^{-x^2 + C} = \boxed{e^{C}} \cdot e^{-x^2}$$

k

$$\boxed{y = k e^{-x^2}}$$

GENERAL SOLN.

$(k = 0 \rightarrow y = 0)$

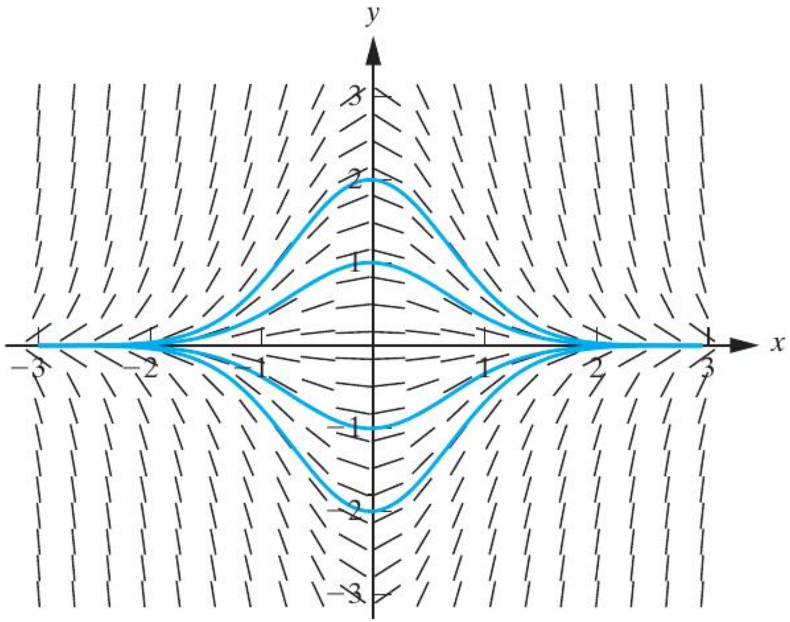


Figure 1.4.2: Slope field and some solution curves for the differential equation $\dfrac{dy}{dx} = -2xy$.

§ 1.6 FIRST-ORDER LINEAR DIFF. EQNS.

DEFINITION 1.6.1

A differential equation that can be written in the form

p(x)

q(x)

$$\cancel{a(x)}\frac{dy}{dx} + \boxed{\frac{b(x)}{a(x)}}y = \boxed{\frac{r(x)}{a(x)}}, \qquad (1.6.1)$$

where $a(x)$, $b(x)$, and $r(x)$ are functions defined on an interval (a, b), is called a **first-order linear differential equation**.

We assume that $a(x) \neq 0$ on (a, b) and divide both sides of (1.6.1) by $a(x)$ to obtain the **standard form**

$$\boxed{\frac{dy}{dx} + p(x)y = q(x),} \qquad (1.6.2)$$

Example 1.6.2 Solve the differential equation

$p(x)$

$$\frac{dy}{dx}+\boxed{\frac{1}{x}}y=e^x, \qquad x>0. \tag{1.6.3}$$

$q(x)$

$$\frac{dy}{dx}+\frac{1}{x}y=e^x \rightarrow \text{MULTIPLY BY } x$$

$$\Rightarrow \boxed{x\frac{dy}{dx}+y}=xe^x$$

????

$$\frac{d}{dx}(xy) \underset{\text{(PRODUCT RULE)}}{=} \underbrace{\left(\frac{dx}{dx}\right)}_{1}y+x\left(\frac{dy}{dx}\right)=\boxed{y+x\frac{dy}{dx}}$$

$$\frac{d}{dx}(xy) = xe^x$$

$\Downarrow$ ANTIDERIVATIVES

$$xy = \boxed{\int xe^x\,dx} + C$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x$$

$u = x \Rightarrow du = 1$

$dv = e^x\,dx \Rightarrow v = e^x$

$$xy = xe^x - e^x + C$$

$$\Rightarrow y = e^x - \frac{e^x}{x} + \frac{C}{x}$$ [DIVIDE BY x]

"INTEGRATING FACTOR"

$$\boxed{\frac{dy}{dx} + p(x)\,y = q(x)} \quad \text{— I}$$

WANT TO MULTIPLY BY $I(x)$ SUCH THAT THE LEFT HAND SIDE BECOMES

$$\frac{d}{dx}\left(I(x)\,y\right)$$

$$\frac{d}{dx}\left(I(x)\,y\right) \leftarrow \boxed{I(x)\frac{dy}{dx} + I(x)\,p(x)\,y} = I(x)\,q(x)$$

$$\frac{d}{dx}\left(I(x)\,y\right) = \boxed{\frac{dI}{dx}}\,y + \boxed{I(x)\,\frac{dy}{dx}}$$

$$\boxed{I(x)\,\frac{dy}{dx}} + \boxed{I(x)\,p(x)}\,y$$

FIND: $I(x)$ S.T.

$$\frac{dI}{dx} = I(x)\,p(x)$$

$$\Rightarrow \quad \frac{dI}{I} = p(x)\,dx \quad \Rightarrow \quad \ln I = \int p(x)\,dx$$

$$\boxed{I = e^{\int p(x)\,dx}}$$

$$\frac{d}{dx}\left(I(x)\,y\right) = I(x)\,q(x)$$

$$\downarrow$$

$$I(x)\,y = \int I(x)\,q(x)\,dx + C$$

$$\boxed{y = \frac{1}{I(x)}\int I(x)\,q(x)\,dx + \frac{C}{I(x)}}$$

$$\frac{dy}{dx} + \frac{1}{x}y = e^x$$

$$p(x) = 1/x$$

$$\int p(x)\,dx = \int \frac{1}{x}\,dx = \ln x$$

$$I(x) = e^{\int p(x)\,dx} = e^{\ln x}$$

$$= x$$

NOTE: DON'T MEMORIZE!

Example 1.6.3 Solve the initial-value problem

$$\frac{dy}{dx} + xy = xe^{x^2/2}, \qquad y(0) = 1.$$

$e^{x^2/2}$

$p(x) = x$

$$\int p(x)\,dx = \frac{x^2}{2}$$

$$I(x) = e^{\int p(x)\,dx} = e^{x^2/2}$$

$$\frac{d}{dx}\left(I(x)\,y\right) = \boxed{e^{x^2/2}\frac{dy}{dx} + xe^{x^2/2}\,y} = x e^{x^2}$$

$$\frac{d}{dx}\left(e^{x^2/2}\,y\right) = \frac{d}{dx}\left(e^{x^2/2}\right)y + e^{x^2/2}\frac{dy}{dx}$$

$$= x\,e^{x^2/2}\,y + e^{x^2/2}\frac{dy}{dx}$$