MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
M – 9:00 PM – 10:00 PM (ET)
F – 3:00 PM – 4:00 PM (ET)
(N.B.: THIS WEEK – T, 9:00 PM)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1-4 ARE UPLOADED.

2. WW 01 - WAS DUE SATURDAY (2nd JULY) AT 11:00PM ET
   WW 02 - IS DUE ~~TUE~~ WED (6th JULY) AT 11:00 PM ET
   WW 03 - IS DUE SATURDAY (9th JULY) AT 11:00 PM ET

3. THIS WEEK: OFFICE HOURS ON TUES (9:00 PM ET)

4. EXAM SLOTS: 8 PM ET (FINAL), 9 PM ET (MIDTERMS)
   FORM FOR WHICH SLOT.

5. REMINDER: PLEASE KEEP VIDEOS ON, IF POSSIBLE!

§2.1 MATRICES: DEFN & NOTATION (CONT'D)

SQUARE MATRICES

$m = n$

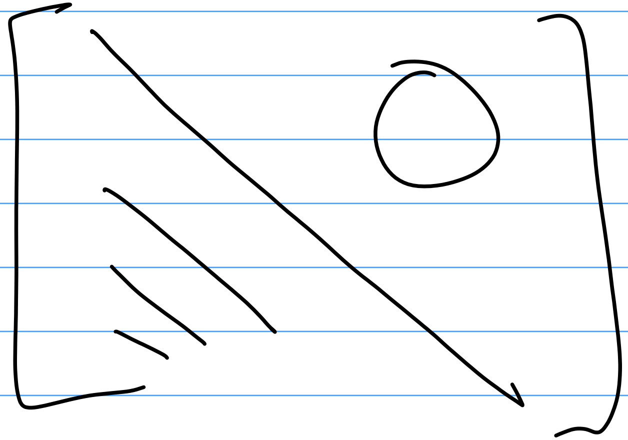
DIAGONAL OF A SQUARE MATRIX

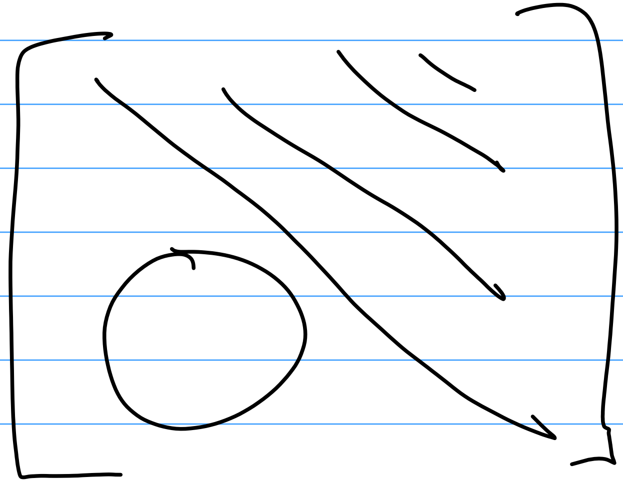
$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = [a_{ij}] \quad \begin{matrix} 1 \le i \le n \\ 1 \le j \le n \end{matrix}$$

$= A$

$$\operatorname{Tr} A = \sum_j a_{jj}$$

DIAGONAL = LOWER Δ-LAR + UPPER Δ-LAR





$$\operatorname{diag}(d_1, \dots, d_n) = \begin{bmatrix} d_1 & & & \huge 0 \\ & d_2 & & \\ & & \ddots & \\ \huge 0 & & & d_n \end{bmatrix}$$

$$\operatorname{diag}(1,2) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \operatorname{diag}(2,1)$$

e.g. $$\operatorname{diag}(-5, 0, -9, 4) = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(SKEW-) SYMMETRIC MATRICES

DEFINITION 2.1.13

1. A square matrix A satisfying $A^T = A$ is called a **symmetric matrix**.
2. If $A = [a_{ij}]$, then we let $-A$ denote the matrix with elements $-a_{ij}$. A square matrix A satisfying $A^T = -A$, is called a **skew-symmetric** (or **anti-symmetric**) **matrix**.

RECALL: $A^T \rightarrow$ FLIPPING ALONG DIAGONAL

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 5 \\ -1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 0 \\ 3 & 5 & 0 \end{bmatrix}$$

$$(A^T)^T = A$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^T = -A$$

$$\left(-A := [-a_{ij}] ; \ A = [a_{ij}] \right)$$

§ 2.2 MATRIX ALGEBRA

1. ADDITION
2. SUBTRACTION
3. SCALAR MULTIPLICATION
4. MATRIX MULTIPLICATION *

1. ADDITION

DEFINITION 2.2.1

If A and B are both $m \times n$ matrices, then we define **addition** (or the **sum**) of A and B, denoted by $A + B$, to be the $m \times n$ matrix whose elements are obtained by adding corresponding elements of A and B. In index notation, if $A = [a_{ij}]$ and $B = [b_{ij}]$, then $A + B = [a_{ij} + b_{ij}]$.

N.B. : SAME DIMENSION !

$B + A = A + B$

$$A = \begin{bmatrix} -2 & 1 & 0 & 4 \\ -1 & -3 & 2 & 2 \end{bmatrix}$$

$2 \times 4 \leftarrow$ $$B = \begin{bmatrix} 6 & 3 & -3 & 3 \\ -3 & 3 & -2 & -5 \end{bmatrix}$$

$2 \times 3 \leftarrow$ $$C = \begin{bmatrix} 3 & 0 & 7 \\ -4 & 1 & 1 \end{bmatrix},$$

$A + B$?

$B + C$?

$A + C$?

$\left.\begin{matrix} B + C \\ A + C \end{matrix}\right\} \to$ DON'T EXIST

$$A + B = \begin{bmatrix} 4 & 4 & -3 & 7 \\ -4 & 0 & 0 & -3 \end{bmatrix}$$

PROPERTIES

Properties of Matrix Addition: If A and B are both $m \times n$ matrices, then

$$A + B = B + A \qquad \text{(Matrix addition is commutative)},$$
$$A + (B + C) = (A + B) + C \qquad \text{(Matrix addition is associative)}.$$

PF IN THE BOOK

3. $\boxed{\text{SCALAR}}$ MULTIPLICATION

$\hookrightarrow \mathbb{R}$ (REAL NUMBER)

DEFINITION 2.2.3

If A is an $m \times n$ matrix and s is a scalar, then we let sA denote the matrix obtained by multiplying every element of A by s. This procedure is called **scalar multiplication**. In index notation, if $A = [a_{ij}]$, then $sA = [sa_{ij}]$.

$$sA = \left[\begin{array}{c} \text{EVERY COMPONENT OF} \\ \text{A IS MULTIPLIED BY s} \end{array} \right]$$

$$A = [a_{ij}] \quad , \quad sA = [sa_{ij}]$$

$$A = \begin{bmatrix} -2 & 1 & 0 & 4 \\ -1 & -3 & 2 & 2 \end{bmatrix}, \quad 1A = \begin{bmatrix} -2 & 1 & 0 & 4 \\ -1 & -3 & 2 & 2 \end{bmatrix} = A$$

$$1A \; (:= A)$$

$$B = \begin{bmatrix} 6 & 3 & -3 & 3 \\ -3 & 3 & -2 & -5 \end{bmatrix}$$

$$(-7)B$$

$$C = \begin{bmatrix} 3 & 0 & 7 \\ -4 & 1 & 1 \end{bmatrix}, \quad 0C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$0C \; (:= 0)$$

MATRIX 0

$$2A$$

$$2A = \begin{bmatrix} 2(-2) & 2(1) & 2(0) & 2(4) \\ 2(-1) & 2(-3) & 2(2) & 2(2) \end{bmatrix} = \begin{bmatrix} -4 & 4 & 0 & 8 \\ -2 & -6 & 4 & 4 \end{bmatrix}$$

$$-7B = \begin{bmatrix} -42 & -21 & 21 & -21 \\ 21 & -21 & 14 & 35 \end{bmatrix}$$

Properties of Scalar Multiplication: For any scalars s and t, and for any matrices A and B of the same size,

$$
\begin{aligned}
1A &= A && \text{(Unit property),}\\
s(A+B) &= sA + sB && \text{(Distributivity of scalars over matrix addition),}\\
(s+t)A &= sA + tA && \text{(Distributivity of scalar addition over matrices),}\\
s(tA) = (st)A &= (ts)A = t(sA) && \text{(Associativity of scalar multiplication).}
\end{aligned}
$$

FOLLOW FROM DISTRIBUTITY IN $\mathbb{R}$

$(s, t) \rightarrow$ NUMBERS

$A \rightarrow$ MATRIX

2\. SUBTRACTION

DEFINITION 2.2.5

MATRIX ADDITION

If A and B are both $m \times n$ matrices, then we define **subtraction** of these two matrices by

SCALAR MULTIPLICATION

$$A - B = A + (-1)B.$$

In index notation $A - B = [a_{ij} - b_{ij}]$. That is, we subtract corresponding elements.

NOTE : NOT COMMUTATIVE ! → $A - B \neq B - A$

NOTE : SAME DIM !

$$A = \begin{bmatrix} \boxed{-2} & \boxed{1} & \boxed{0} & \boxed{4} \\ -1 & -3 & 2 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} \boxed{6} & \boxed{3} & \boxed{-3} & 3 \\ -3 & 3 & -2 & -5 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 0 & 7 \\ -4 & 1 & 1 \end{bmatrix},$$

$A - B$

$A - A$

$\boxed{C - A} \to$?? DOESN'T EXIST!

$-2-6 \quad 1-3 \quad 0-(-3) \quad 4-3$

$$A - B = \begin{bmatrix} -8 & -2 & 3 & 1 \\ 2 & -6 & 4 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 1 & 0 & 4 \\ -1 & -3 & 2 & 2 \end{bmatrix}$$

$(-2) - (-2) = 0$

$1 - 1 = 0$

$0 - 0 = 0$

$$A - A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ZERO MATRIX $O_{m \times n}$

$$O_{m \times n} = \begin{array}{cl} \overbrace{\hphantom{\begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}}}^{n\text{-COLUMNS}} & \\ \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} & \left.\vphantom{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}\right\} m \text{ ROWS} \end{array}$$

$a_{ij} + 0 = a_{ij}$ $a_{ij} - a_{ij} = 0$ $0 a_{ij} = 0$

Properties of the Zero Matrix: For all matrices A and the zero matrix of the same size, we have

$$\boxed{A + 0 = A,} \qquad \boxed{A - A = 0,} \qquad \text{and} \qquad \boxed{0A = 0.}$$

MATRIX $(0_{m \times n})$

$m \times n \to$ DIM A

Note that in the last property here, the zero on the left side of the equation is a scalar, while the zero on the right side of the equation is a matrix.

NUMBER

$A \to m \times n$ MATRIX

4. MATRIX MULTIPLICATION

→ VERY COMPLEX!

→ WILL DO IN STAGES

SAME SIZE!

CASE I: ROW n-VECTOR X COLUMN n-VECTOR

FIRST NEED TO DEFINE DOT PRODUCT.

$\vec{a}\ ,\ \vec{b} \rightarrow$ VECTORS (COLUMN OR ROW)

$\vec{a} = [a_1 \ \cdots \ a_n]$

$\vec{b} = [b_1 \ \cdots \ b_n]$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n .$$

DOT PRODUCT

DOT (VECT, VECT) = SCALAR

$$\vec{a} \cdot \vec{b} = \sum_{j=1}^{n} a_j b_j = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$\vec{a} = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

$$\vec{a} \cdot \vec{b} = 1 \times 1 + 3 \times 0 + 2 \times (-1)$$

$$= 1 + 0 - 2 = -1$$

COLUMN

$a \; x$ = 1 × 1 MATRIX GIVEN BY $\vec{a} \cdot \vec{x}^T$

ROW

$$\mathbf{ax} = [a_1 \quad a_2 \quad \dots \quad a_n]\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = [a_1x_1 + a_2x_2 + \cdots + a_nx_n].$$

(ROW) $\vec{a}$

(COLUMN) $\vec{x} \longrightarrow \vec{x}^T \longrightarrow \vec{a} \cdot \vec{x}^T \longrightarrow \left[\sum_j a_j x_j\right]$

(SCALAR)

$$\mathbf{a} = \begin{bmatrix} -8 & 3 & -1 & 2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 7 \\ -5 \end{bmatrix},$$

$$\vec{a}\,\vec{x} = \begin{bmatrix} -8 & 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 7 \\ -5 \end{bmatrix} = \begin{bmatrix} (-8)(1) + (3)(4) + (-1)(7) \\ + 2(-5) \end{bmatrix}$$

$$= \begin{bmatrix} -13 \end{bmatrix}$$

CASE II : (m×n MATRIX) × (COLUMN n-VECTOR)

¡¡¡ NOTE !!! → DIMENSIONS.

$\vec{a}_j \to$ ROW n-VECTORS

COLUMN m-VECTOR

$$\underset{m\times n}{A}\,\overset{n\times 1}{\vec{x}} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{a}_1\,\vec{x} \\ \vec{a}_2\,\vec{x} \\ \vdots \\ \vec{a}_m\,\vec{x} \end{bmatrix} = \begin{bmatrix} \sum_j a_{1j}\,x_j \\ \sum a_{2j}\,x_j \\ \vdots \\ \sum a_{mj}\,x_j \end{bmatrix}$$

$A = m \times n$ MATRIX

$$= [a_{ij}]_{\substack{1 \le i \le m \\ 1 \le j \le n}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \to \vec{a}_1 \\ \to \vec{a}_2 \\ \\ \to \vec{a}_m \end{matrix}$$

$$A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix} \to m \text{ ROW } n\text{-VECTOR}$$

$A\vec{x}$

$$\begin{array}{c} \\ \\ \\ \text{Row } i \\ \\ \\ \end{array}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \vdots \\ \\ x_n \end{bmatrix} = \begin{bmatrix} (A\mathbf{x})_1 \\ (A\mathbf{x})_2 \\ \vdots \\ (A\mathbf{x})_i \\ \vdots \\ (A\mathbf{x})_m \end{bmatrix}\begin{array}{l} \\ \\ i\text{th element of } A\mathbf{x} \\ \\ \\ \\ \end{array}$$

$m \times n$

$(m, n = 3)$

$$A = \begin{bmatrix} \boxed{-3 \quad 1 \quad -2} \\ \boxed{0 \quad 5 \quad -2} \\ \boxed{-4 \quad -2 \quad 5} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}$$

n

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}$$

$$\vec{a}_1 \vec{x} = \begin{bmatrix} -3 & 1 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}$$

$$= 6 + 1 + (-12) = -5$$

$$\vec{a}_2 \vec{x} = \begin{bmatrix} 0 & 5 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix} = 0 + 5 - 12 = -7$$

$$\vec{a}_3 \vec{x} = \begin{bmatrix} -4 & -2 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix} = 36$$

$$A \vec{x} = \begin{bmatrix} \vec{a}_1 \vec{x} \\ \vec{a}_2 \vec{x} \\ \vec{a}_3 \vec{x} \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 36 \end{bmatrix}$$

$\downarrow$

m-VECTOR

$\downarrow$

COLUMN 3-VECTOR

$$A = \begin{bmatrix} -3 & 1 & -2 \\ 0 & 5 & -2 \\ -4 & -2 & 5 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & -2 \\ 0 & 5 & -2 \\ -4 & -2 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 36 \end{bmatrix}$$

$$(-3)(-2) + (1)(1) + (-2)(6)$$

$$\boxed{} = (0)(-2) + (5)(1) + (-2)(6)$$

BREAK TILL

10:15 AM

CASE 3 : $(m \times \boxed{n}$ MATRIX$) \times (\boxed{n} \times p$ MATRIX$)$

(MOST GENERAL)

$$\underset{m \times n}{\underset{\sim}{A}} \quad \overset{n \times p}{\underset{\sim}{B}}$$

$$B = [\vec{b}_1 \quad \cdots \quad \vec{b}_p]$$

$$A[\vec{b}_1 \quad \cdots \quad \vec{b}_p] = [A\vec{b}_1 \quad \cdots \quad A\vec{b}_p]$$

$$AB = [A\vec{b}_1 \quad \cdots \quad A\vec{b}_p] = \begin{bmatrix} \vec{a}_1 \cdot \vec{b}_1 & \cdots & \vec{a}_1 \cdot \vec{b}_p \\ \vec{a}_2 \cdot \vec{b}_1 & \ddots & \vdots \\ \vdots & & \vdots \\ \vec{a}_m \cdot \vec{b}_1 & \cdots & \vec{a}_m \cdot \vec{b}_p \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_n \end{bmatrix}, \quad B = [\vec{b}_1 \quad \cdots \quad \vec{b}_p]$$

$$C_{m \times p} = A_{m \times n} \; B_{n \times p}$$

$C = [c_{ij}]$

$A = [a_{ij}]$

$B = [b_{ij}]$

DEFINITION 2.2.13

If $A = [a_{ij}]$ is an $m \times n$ matrix, $B = [b_{ij}]$ is an $n \times p$ matrix, and $C = AB$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad 1 \leq i \leq m, \quad 1 \leq j \leq p. \tag{2.2.3}$$

This is called the **index form** of the matrix product.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \vec{a}_i \cdot \vec{b}_j$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots a_{ik} b_{kj}$$

OUTPUT

$2 \times 3 \qquad 3 \times 2$

$m = 2$
$n = 3$
$p = 2$

If $A = \begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 \\ 3 & -1 \\ -9 & 2 \end{bmatrix}$, determine AB.

$$\begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & 6 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 3 & -1 \\ -9 & 2 \end{bmatrix} = \begin{bmatrix} \boxed{-16} & \boxed{3} \\ \boxed{-76} & \boxed{20} \end{bmatrix}$$

$\Box = (-2)(-4) + (1)(3) + 3(-9) = -16$

$\Box = -16 - 6 - 54$
$= -76$

$\Box = 4 + 2 + 12 = 20$

$\Box = (-2) + (-1) + 6$

NOTE 1 : WHEN IS IT DEFINED?

$$A_{m \times n} \cdot B_{n \times p}$$

NOTE 2 : WHAT ARE OUTPUT DIMENSIONS?

$$m \times p$$

If $A = \begin{bmatrix} 6 \\ -4 \\ 0 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \end{bmatrix}$, determine AB.

4×1 $\quad$ 1×2

BA → NOT DEFINED

OUTPUT DIM $= 4 \times 2$

$$AB = \begin{bmatrix} 6 \\ -4 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \end{bmatrix} = \begin{bmatrix} \boxed{6} & \boxed{-18} \\ \boxed{-4} & 12 \\ 0 & 0 \\ 1 & -3 \end{bmatrix}$$

$A\ B \rightarrow$ DEFINED

$B\ A \rightarrow$ NOT DEFINED

$$\begin{array}{cc} 1\times 2 & \\ A = \begin{bmatrix} -2 & -1 \end{bmatrix} \end{array} \text{ and } \begin{array}{c} 2\times 3 \\ B = \begin{bmatrix} 4 & -6 & -3 \\ 2 & 6 & 3 \end{bmatrix}. \end{array}$$

$$\begin{bmatrix} -2 & -1 \end{bmatrix} \begin{bmatrix} 4 & -6 & -3 \\ 2 & 6 & 3 \end{bmatrix} = \begin{bmatrix} -10 & 6 & 3 \end{bmatrix}$$

PROPERTIES

Theorem 2.2.18 If A, B, and C have appropriate dimensions for the operations to be performed, then

$$A(BC) = (AB)C \qquad \text{(Associativity of matrix multiplication),} \qquad \boxed{(2.2.4)}$$

$$A(B + C) = AB + AC \qquad \text{(Left distributivity of matrix multiplication),} \qquad (2.2.5)$$

$$(A + B)C = AC + BC \qquad \text{(Right distributivity of matrix multiplication).} \qquad (2.2.6)$$

$A \rightarrow m \times n$

$B \rightarrow n \times p$

$C \rightarrow p \times q$

BC & AB MAKING SENSE

$$(AB)C \stackrel{?}{=} A(BC)$$

$A \to m \times n$

$B \to n \times p$

$C \to p \times q$

$A_{m \times n} \; B_{n \times p}$

$B_{n \times p} \; C_{p \times q}$

$$\dim(AB) = m \times p$$

$$\dim(BC) = n \times q$$

$$(AB)C \to (AB)_{m \times p} \; C_{p \times q} = [(AB)C]_{m \times q}$$

$$A(BC) \to A_{m \times n} \; (BC)_{n \times q} = [A(BC)]_{m \times q}$$

$\boxed{m \neq n}$

$AB, BA \rightarrow$ DEFINED

$A \rightarrow m \times n$

$B \rightarrow n \times m$

$A_{m\times n} B_{n\times m} \rightarrow m \times m$ & $B_{n\times m} A_{m\times n} \rightarrow n \times n$

$\underbrace{(AB)}_{m \times m} \neq \underbrace{(BA)}_{n \times n}$

WHAT IF $m = n$?

$A, B \to n \times n$

$AB \quad , \quad BA$

If $A = \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 2 \\ 3 & -2 \end{bmatrix}$, find AB and BA.

$$AB = \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -18 & 8 \\ 22 & -12 \end{bmatrix}$$

$$AB \neq BA$$

$$BA = \begin{bmatrix} -5 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -17 & 13 \\ 13 & -11 \end{bmatrix}$$

MATRIX MULT. IS
NOT COMMUTATIVE !!!

(n-DIM) IDENTITY MATRIX

$(A)_{m\times n} \, (I)_{n\times n}$

$(I)_{m\times m} A_{m\times n}$

DEFINITION 2.2.21

The elements of I_n can be represented by the **Kronecker delta symbol**, δ_{ij}, defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Then,

$$I_n = [\delta_{ij}].$$

I

$I_n = \mathrm{diag}(\underbrace{1, \dots, 1}_{n})$

$$I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

$A \to 2\times 3$

$$A = \begin{bmatrix} -1 & 4 & 0 \\ 7 & -3 & 2 \end{bmatrix},$$

$$\boxed{AI = IA = A}$$

$$AI = \begin{bmatrix} -1 & 4 & 0 \\ 7 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 0 \\ 7 & -3 & 2 \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 & 0 \\ 7 & -3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 0 \\ 7 & -3 & 2 \end{bmatrix} = A$$

Properties of the Identity Matrix:

1. $A_{m\times n} I_n = A_{m\times n}.$
2. $I_m A_{m\times p} = A_{m\times p}.$

$AI = A$

$IA = A$

RECALL : TRANSPOSE

$\downarrow$

FLIP ALONG THE DIAGONAL

Theorem 2.2.23 Let A and C be $m \times n$ matrices, and let B be an $n \times p$ matrix. Then

1. $(A^T)^T = A$.
2. $(A + C)^T = A^T + C^T$.
3. $(AB)^T = B^T A^T$.

$A \to m \times n$

$C \to m \times n$

$$\underbrace{(A + C)^T}_{n \times m} = \overbrace{A^T}^{n \times m} + \overbrace{C^T}^{n \times m}$$

$A = [a_{ij}]$

$C = [c_{ij}]$

$A + C = [a_{ij} + c_{ij}]$, $(A + C)^T = [a_{ji} + c_{ji}]$

$A^T = [a_{ji}]$

$C^T = [c_{ji}]$

$A^T + C^T = [a_{ji} + c_{ji}]$

$$(AB)^T = B^T A^T$$

$$A = [a_{ik}]$$

$$B = [b_{kj}]$$

$$AB$$

$$\left[(AB)_{ij}\right]^T$$