MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:

T ~~M~~ – 9:00 PM – 10:00 PM (ET)

F – 3:00 PM – 4:00 PM (ET)

LECTURES:

9:00 AM – 11:15 AM (ET)

M, T, W, R

Zoom ID:

979-4693-0650

COURSE WEBPAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1-5 ARE UPLOADED.

2. WW 01 – WAS DUE SATURDAY (2nd JULY) AT 11:00PM ET
   WW 02 – IS DUE ~~TUE~~ WED (6th JULY) AT 11:00 PM ET
   WW 03 – IS DUE SATURDAY (9th JULY) AT 11:00 PM ET

3. PLEASE FILL OUT FORM FOR MIDTERM 1 SLOTS.

4. REMINDER: PLEASE KEEP VIDEOS ON, IF POSSIBLE!

5. OFFICE HOURS HAVE CHANGE (c.f. PREVIOUS)

§ 2.3 TERMINOLOGY FOR SYSTEMS OF LINEAR EQUATIONS

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} & b_m \end{array}\right]$$

DEFINITION 2.3.1

The general $m \times n$ system of linear equations is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned} \tag{2.3.1}$$

where the **system coefficients** a_{ij} and the **system constants** $\boxed{b_j}$ are given scalars and $\boxed{x_1, x_2, \dots, x_n}$ denote the unknowns in the system. If $b_i = 0$ for all i, then the system is called **homogeneous**; otherwise it is called **nonhomogeneous**.

$$\begin{array}{ccc} x_1, & \cdots & x_n \\ \downarrow & & \downarrow \\ c_1, & \cdots, & c_n \end{array}$$

DEFINITION 2.3.2

By a **solution** to the system (2.3.1) we mean an ordered n-tuple of scalars, $(c_1, c_2, \ldots, c_n)$, which, when substituted for $x_1, x_2, \ldots, x_n$ into the left-hand side of system (2.3.1), yield the values on the right-hand side. The set of all solutions to system (2.3.1) is called the **solution set** to the system.

SET OF SOLUTIONS IS INFINITE!

Verify that for all real numbers a and b, the 4-tuple $(\overbrace{24+5a-10b}^{x_1}, \underbrace{-8-4a+3b}_{x_2}, \overset{x_3}{a}, \underset{x_4}{b})$ satisfies the system

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 8, \quad \text{— I}$$
$$2x_1 + 5x_2 + 10x_3 + 5x_4 = 8. \quad \text{— II}$$

PLUG IN

$x_1 = 24 + 5a - 10b$
$x_2 = -8 - 4a + 3b$
$x_3 = a$
$x_4 = b$

✓ I: $24 + 5\cancel{a} - \cancel{10b} + 2(-8 - \cancel{4a} + \cancel{3b}) + 3\cancel{a} + \cancel{4b} = 8$

✓ II: $2(24 + 5a - 10b) + 5(-8 - 4a + 3b) + 10a + 5b$

$= 48 + \cancel{10a} - \cancel{20b} - 40 - \cancel{20a} + \cancel{15b} + \cancel{10a} + \cancel{5b} = 8$

<u>IMPORTANT QUESTIONS:</u>

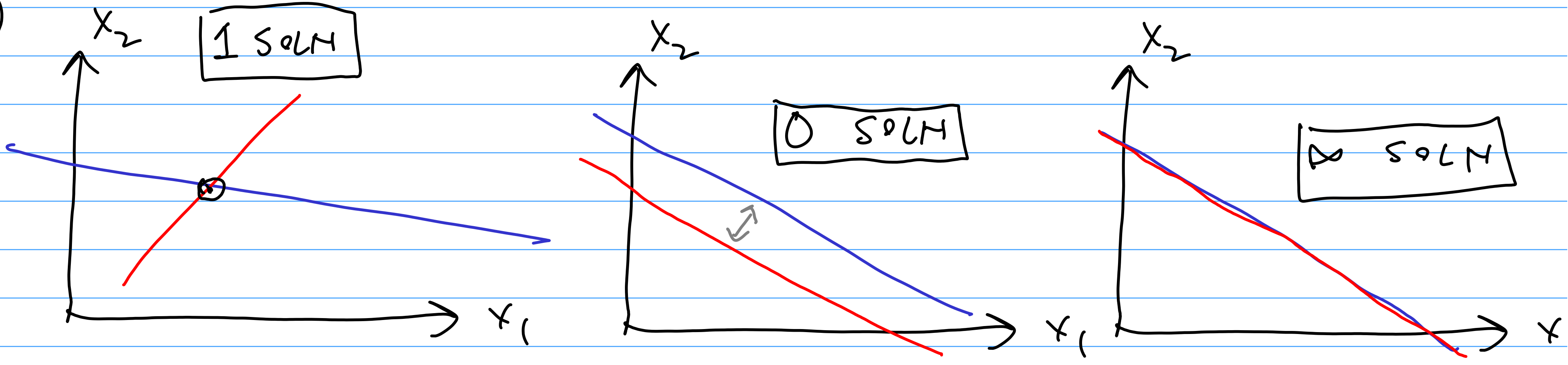
1. ARE THERE ANY SOLUTIONS?
2. IF YES, HOW MANY?
3. HOW DO WE FIND THEM?

MODEL CASE : SYSTEM OF 2 EQUATIONS IN 2 VARIABLES.

$$\boxed{a_{11}x_1 + a_{12}x_2 = b_1} \quad \text{I}$$

$$\boxed{a_{21}x_1 + a_{22}x_2 = b_1} \quad \text{II}$$

(a)



DEFINITION 2.3.4

A system of equations that has at least one solution is said to be **consistent**, whereas a system that has no solution is called **inconsistent**.

DEFINITION 2.3.5

Naturally associated with the system (2.3.1) are the following two matrices:

1. The **matrix of coefficients** $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$.

2. The **augmented matrix** $A^{\#} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$.

   The bar just to the left of the rightmost column is useful for visually separating the matrix of coefficients from the constants given on the right side of the linear system.

Write the system of equations with the following augmented matrix:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array}$$

$$\left[\begin{array}{cccc|c} -2 & 0 & 5 & -1 & 6 \\ 4 & -1 & 2 & 2 & -2 \\ -7 & -6 & 0 & 4 & -8 \end{array}\right].$$

$x_1 \quad x_2 \quad x_3$

$$(-2)x_1 + 0x_2 + 5x_3 + (-1)x_4 = 6$$

$$4x_1 + (-1)x_2 + 2x_3 + 2x_4 = -2$$

$$(-7)x_1 + (-6)x_2 + 0x_3 + 4x_4 = -8$$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m,
\end{aligned}
$$

$A =$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

A $\rightarrow$ COEFFICIENT MATRIX

x $\rightarrow$ VECTOR OF VARIABLES

b $\rightarrow$ VECTOR OF COEFFICIENTS

SYSTEM OF LINEAR EQUATIONS $\longleftrightarrow$ MATRIX / VECTOR $\Updownarrow$ AUGMENTED MATRIX $\longleftrightarrow$ MATRIX EQ $A\vec{x} = \vec{b}$

NOTATION: $\mathbb{R}^n \rightarrow$ TUPLES $(x_1, \dots, x_n)$

$(\mathbb{R}^2 \longrightarrow (x, y))$; $(\mathbb{R}^3 \longrightarrow (x, y, z))$

NOTE: $(x_1, \dots, x_n) \cong \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \cong \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$\mathbb{R} \to$ SET OF REAL #s.

OPERATIONS ON $\mathbb{R}^n$

1. ADDITION
2. SUBTRACTION

$$(x_1, \dots, x_n) \pm (y_1, \dots, y_n) = (x_1 \pm y_1, \dots, x_n \pm y_n)$$

3. SCALAR MULTIPLICATION $\quad c(x_1, \dots, x_n) = (cx_1, \dots, cx_n)$

4. DOT PRODUCT $\longrightarrow (x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = \sum_{j=1}^{n} x_j y_j$

$A\vec{x} = \vec{b} \longrightarrow$ SOLUTION SET

$S \subseteq \mathbb{R}^n$, $S = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{b} \}$

$A \rightarrow$ FIXED MATRIX

$b \rightarrow$ FIXED VECTOR

§2.4 ROW ECHELON MATRICES AND ELEMENTARY ROW OPERATIONS

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 6, \\
x_2 - 4x_3 &= -4, \\
x_3 &= 3.
\end{aligned}
$$

BACK SUBSTITUTION

$x_2 - 4(3) = -4 \Rightarrow x_2 = 8$

$x_1 + 8 + 3 = 6 \Rightarrow x_1 = -5$

$(x_1, x_2, x_3) = (-5, 8, 3)$

AUGMENTED MATRIX

ROW ECHELON FORM $\leftarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$A \rightarrow$ UPPER TRIANGULAR

DEFINITION 2.4.1

An $m \times n$ matrix is called a **row-echelon matrix** if it satisfies the following three conditions:

1. If there are any rows consisting entirely of zeros, they are grouped together at the bottom of the matrix.
2. The first nonzero element in any nonzero row[4] is a 1 (called a **leading 1**).
3. The leading 1 of any row below the first row is to the right of the leading 1 of the row above it.

$$\begin{bmatrix} 1 & -8 & -3 & 7 \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ \boxed{0 \quad 0 \quad 0} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -3 & -6 & 5 & 7 \\ 0 & 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

YES NO NO YES

$$x_1 - x_2 = 5 \quad - \mathrm{I}$$

$$x_1 + x_2 = 1 \quad - \mathrm{II}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 1 & 1 & 1 \end{array}\right]$$

$$\mathrm{III} := \frac{\mathrm{II} - \mathrm{I}}{2}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -2 \end{array}\right]$$

$$\mathrm{III}: \quad \frac{x_1 - x_1}{2} + \frac{x_2 - (-x_2)}{2} = \frac{1-5}{2}$$

$$0 x_1 + x_2 = -2$$

$$\mathrm{II} = \mathrm{I} + 2\,\mathrm{III}$$

SET OF SOLN OF I, II $\Longleftrightarrow$ SET OF SOLN OF I, III

$$\left.\begin{aligned} x_1 - x_2 &= 5 \\ 0x_1 + x_2 &= -2 \end{aligned}\right\} \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -2 \end{array}\right]$$

$$x_1 + x_2 = 3 \quad \longleftrightarrow \quad 2x_1 + 2x_2 = 6$$

1. Permute equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of one equation to another equation.

DOES NOT CHANGE SOLN SET!

WHAT DOES THIS CORRESPOND TO FOR THE AUGMENTED MATRIX ?

ELEMENTARY ROW OPERATIONS

1. Permute rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of one row to another row.

$$P_{ij} \quad \begin{array}{c} i \\ j \end{array}\begin{bmatrix} \; \\ \; \end{bmatrix}$$

1. P_{ij}: **P**ermute the ith and jth rows of A.
2. $\mathrm{M}_i(k)$: **M**ultiply every element of the ith row of A by a nonzero scalar k.
3. $\mathrm{A}_{ij}(k)$: **A**dd to the elements of the jth row of A the scalar k times the corresponding elements of the ith row of A.

$A_{ij}(k) \rightarrow$ ROW $i \times k$ IS ADDED TO ROW j

$$\begin{array}{c} i \end{array}\begin{bmatrix} k \; \cdots k \; \cdots \end{bmatrix}$$

ROW EQUIVALENCE

DEFINITION 2.4.5

Let A be an $m \times n$ matrix. Any matrix obtained from A by a finite sequence of elementary row operations is said to be **row-equivalent** to A.

$$A \xrightarrow{P_{23}} A' \xrightarrow{M_5(11)} A'' \longrightarrow \cdots \longrightarrow B$$

NOTE: EROs ARE REVERSIBLE!

Theorem 2.4.6 Systems of linear equations with row-equivalent augmented matrices have the same solution sets.

Theorem 2.4.7 Every matrix is row-equivalent to a row-echelon matrix.

RECALL: ROW ECHELON MATRICES CAN BE SOLVED BY BACK SUBSTITUTION!

P_{ij}

$M_j(k)$

$A_{ij}(k)$

$$\begin{bmatrix} 4 & -3 & 6 & 2 \\ 1 & -3 & 6 & 5 \\ -2 & 3 & -8 & 6 \end{bmatrix}$$

$$\downarrow P_{12}$$

$$\begin{bmatrix} 1 & -3 & 6 & 5 \\ \underset{\sim}{4} & -3 & 6 & 2 \\ \underset{\sim}{-2} & 3 & -8 & 6 \end{bmatrix} \xrightarrow{A_{32}(2)} \begin{bmatrix} 1 & -3 & 6 & 5 \\ 0 & 3 & -10 & 14 \\ -2 & 3 & -8 & 6 \end{bmatrix}$$

2xROW 3 + ROW 2 $\rightarrow$ ROW2

P_{ij}

$M_j(k)$

$A_{ij}(k)$

$$\begin{bmatrix} 1 & -3 & 6 & 5 \\ 0 & 3 & -10 & 14 \\ -2 & 3 & -8 & 6 \end{bmatrix}$$

$$\Big\downarrow A_{13}(2)$$

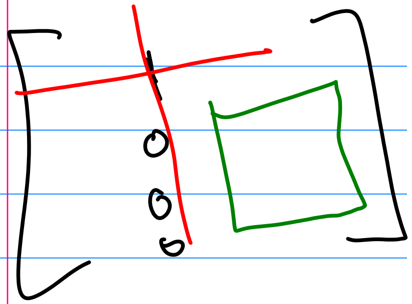
$$\begin{bmatrix} 1 & -3 & 6 & 5 \\ 0 & 3 & -10 & 14 \\ 0 & -3 & 4 & 16 \end{bmatrix} \xrightarrow{A_{23}(1)} \begin{bmatrix} 1 & -3 & 6 & 5 \\ 0 & 3 & -10 & 14 \\ 0 & 0 & -6 & 30 \end{bmatrix}$$

$$M_2(1/3) \Big\downarrow M_3(-1/6)$$

$$\begin{bmatrix} 1 & -3 & 6 & 5 \\ 0 & 3 & -19 & 14 \\ 0 & 0 & -6 & 30 \end{bmatrix} \xrightarrow[M_3(-1/6)]{M_2(1/3)} \begin{bmatrix} 1 & -3 & 6 & 5 \\ 0 & 1 & -19/3 & 14/3 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

BREAK
'TIL

10:23 AM



Algorithm for Reducing an $m \times n$ Matrix A to Row-Echelon Form

1. Start with an $m \times n$ matrix A. If $A = 0$, go to (7).
2. Determine the leftmost nonzero column (this is called a **pivot column** and the topmost position in this column is called a **pivot position**).
3. Use elementary row operations to put a 1 in the pivot position.
4. Use elementary row operations to put zeros below the pivot position.
5. If there are no more nonzero rows below the pivot position go to (7), otherwise go to (6).
6. Apply (2)–(5) to the submatrix consisting of the rows that lie below the pivot position.
7. The matrix is a row-echelon matrix.

NOTE: ROW ECHELON FORM IS NOT UNIQUE!

RANK

Theorem 2.4.10 Let A be an $m \times n$ matrix. All row-echelon matrices that are row-equivalent to A have the same number of nonzero rows.

DEFINITION 2.4.11

The number of nonzero rows in any row-echelon form of a matrix A is called the **rank** of A and is denoted $\text{rank}(A)$.

P_{ij}

$M_i(k)$

$A_{ij}(k)$

PIVOT ELEMENT.

Determine $\boxed{\operatorname{rank}(A)} \to 2$ if $A = \begin{bmatrix} 3 & -1 & 4 & 2 \\ 1 & -1 & 2 & 3 \\ 7 & -1 & 8 & 0 \end{bmatrix}$.

$$\begin{bmatrix} \boxed{3} & -1 & 4 & 2 \\ 1 & -1 & 2 & 3 \\ 7 & -1 & 8 & 0 \end{bmatrix} \xrightarrow{P_{12}} \begin{bmatrix} \boxed{1} & -1 & 2 & 3 \\ 3 & -1 & 4 & 2 \\ 7 & -1 & 8 & 0 \end{bmatrix} \xrightarrow{A_{12}(-3)} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & -7 \\ 7 & -1 & 8 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & -7 \\ \boxed{7} & -1 & 8 & 0 \end{bmatrix} \xrightarrow{A_{13}(-7)} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & -7 \\ 0 & 6 & -6 & -21 \end{bmatrix}$$

$$\xrightarrow{M_2(1/2)} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -7/2 \\ 0 & 6 & -6 & -21 \end{bmatrix} \xrightarrow{A_{23}(-6)} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -7/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

RANK = 2 !

REDUCED ROW ECHELON FORM

DEFINITION 2.4.13

An $m \times n$ matrix is called a **reduced row-echelon matrix** if it satisfies the following conditions:

1. It is a row-echelon matrix.
2. Any *column* that contains a leading 1 has zeros everywhere else.

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

NOT A LEADING 1.

Theorem 2.4.15 An $m \times n$ matrix is row-equivalent to a *unique* reduced row-echelon matrix.

$$\begin{bmatrix} \boxed{1} & -1 & -5 & -3 \\ 3 & 2 & -3 & 5 \\ 2 & 0 & -5 & 1 \end{bmatrix} \xrightarrow[A_{13}(-2)]{A_{12}(-3)} \begin{bmatrix} 1 & -1 & -5 & -3 \\ 0 & 5 & 12 & 14 \\ 0 & 2 & 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -5 & -3 \\ 0 & \boxed{5} & 12 & 14 \\ 0 & 2 & 5 & 7 \end{bmatrix} \xrightarrow{A_{32}(-2)} \begin{bmatrix} 1 & -1 & -5 & -3 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 2 & 5 & 7 \end{bmatrix}$$

$A_{21}(1)$, $A_{23}(-2)$

$$\rightarrow \begin{bmatrix} 1 & 0 & -3 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & \boxed{1} & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & \boxed{1} & 7 \end{bmatrix}$$

$$\Big\downarrow \begin{array}{l} A_{32}(-2) \\ A_{31}(3) \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

§ 2.5 GAUSSIAN ELIMINATION

Determine the solution set to

$$-4+0+6=2$$
$$-1+0+6=5$$
$$(-2)(-1)+0-8=-6$$

$$\begin{cases} 4x_1 - 3x_2 + 6x_3 = 2, \checkmark \\ x_1 - 3x_2 + 6x_3 = 5, \checkmark \\ -2x_1 + 3x_2 - 8x_3 = -6. \checkmark \end{cases}$$

$x_1 = -1$
$x_2 = 0$
$x_3 = 1$

$$\left[\begin{array}{ccc|c} 4 & -3 & 6 & 2 \\ 1 & -3 & 6 & 5 \\ -2 & 3 & -8 & -6 \end{array}\right] \xrightarrow{\text{EROs}} \text{ROW ECHELON FORM}$$

$$\left[\begin{array}{ccc|c} 4 & -3 & 6 & 2 \\ 1 & -3 & 6 & 5 \\ -2 & 3 & -8 & -6 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -3 & 6 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$x_1 - 3x_2 + 6x_3 = 5$$
$$x_2 - 2x_3 = -2 \qquad x_2 = -2 + 2x_3 = -2 + 2 = 0$$
$$\boxed{x_3 = 1}$$

$$x_1 - 3(0) + 6(1) = 5$$
$$x_1 + 6 = 5$$
$$x_1 = -1$$

INSIST ON REDUCED ROW ECHELON

Use $\boxed{\text{Gauss-Jordan elimination}}$ to determine the solution set to

$$\begin{aligned} x_1 - x_2 - 5x_3 &= -3, \\ 3x_1 + 2x_2 - 3x_3 &= 5, \\ 2x_1 \qquad - 5x_3 &= 1. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -5 & -3 \\ 3 & 2 & -3 & 5 \\ 2 & 0 & -5 & 1 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$\left.\begin{aligned} x_1 &= 18 \\ x_2 &= -14 \\ x_3 &= 7 \end{aligned}\right\} \rightarrow \text{CHECK}$$

OF SOLNs

Lemma 2.5.3 Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let $A^\#$ denote the augmented matrix of the system. If $\text{rank}(A) = \text{rank}(A^\#) = n$, then the system has a unique solution.

Lemma 2.5.5 Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let $A^\#$ denote the augmented matrix of the system. If $\text{rank}(A) < \text{rank}(A^\#)$, then the system is inconsistent.

Lemma 2.5.7 Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let $A^\#$ denote the augmented matrix of the system and let $r^\# = \text{rank}(A^\#)$. If $r^\# = \text{rank}(A) < n$, then the system has an infinite number of solutions, indexed by $n - r^\#$ free variables.