MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
T – 9:00 PM – 10:00 PM (ET)
F – 3:00 PM – 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1- 6 ARE UPLOADED.

2. WW 02 — WAS DUE WED (6th JULY) AT 11:00 PM ET
   WW 03 - IS DUE SATURDAY (9th JULY) AT 11:00 PM ET
   WW 04 - IS DUE TUESDAY (12th JULY) AT 11:00 PM ET

3. PLEASE FILL OUT FORM FOR MIDTERM 1 SLOTS.

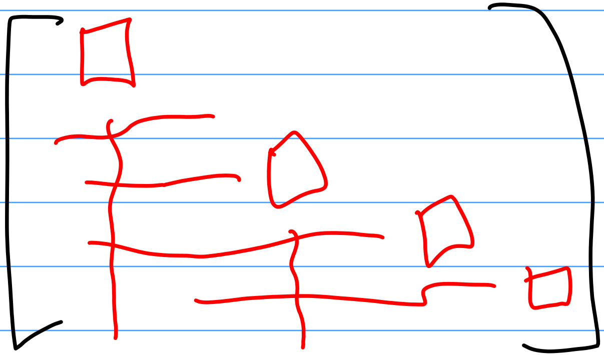
4. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE !

REMARK: $\operatorname{rank} A \leq m, n$ $\qquad A \to m \times n$

$A \longrightarrow A'$ (ROW ECHELON FORM), $\operatorname{rank} A = \#$ of non-zero rows in A'

$\leq \#$ of rows in $A' = m$

$\operatorname{rank} A = \#$ of leading 1s in $A' \leq n$



§ 2.5 GAUSSIAN ELIMINATION
(CONT'D)

INSIST ON REDUCED ROW ECHELON

Use $\boxed{\text{Gauss-Jordan elimination}}$ to determine the solution set to

$$\begin{aligned} x_1 - x_2 - 5x_3 &= -3, \\ 3x_1 + 2x_2 - 3x_3 &= 5, \\ 2x_1 - 5x_3 &= 1. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -5 & -3 \\ 3 & 2 & -3 & 5 \\ 2 & 0 & -5 & 1 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$\left.\begin{aligned} x_1 &= 18 \\ x_2 &= -14 \\ x_3 &= 7 \end{aligned}\right\} \rightarrow \text{CHECK}$$

$$A^{\#} = \left[\begin{array}{ccc|c} 1 & -1 & -5 & -3 \\ 3 & 2 & -3 & 5 \\ 2 & 0 & -5 & 1 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & -1 & -5 \\ 3 & 2 & -3 \\ 2 & 0 & -5 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\text{rank}(A) = 3$, $\text{rank}(A^{\#}) = 3$, $n = 3$

$A^{\#} = [A \mid \vec{b}]$

OF SOLNs

Lemma 2.5.3 Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let $A^{\#}$ denote the augmented matrix of the system. If $\text{rank}(A) = \text{rank}(A^{\#}) = n$, then the system has a unique solution.

Lemma 2.5.5 Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let $A^{\#}$ denote the augmented matrix of the system. If $\text{rank}(A) < \text{rank}(A^{\#})$, then the system is inconsistent.

Lemma 2.5.7 Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let $A^{\#}$ denote the augmented matrix of the system and let $r^{\#} = \text{rank}(A^{\#})$. If $r^{\#} = \text{rank}(A) < n$, then the system has an infinite number of solutions, indexed by $n - r^{\#}$ free variables.

(HEURISTIC: DEGREES OF FREEDOM $= n - r^{\#}$)

$r^{\#}$ → rank A = rank $A^{\#}$ = # OF CONSTRAINTS.

CASE I : $\text{RANK}(A) = \text{RANK}(A^{\#}) = n$

PREVIOUS EXAMPLES ! (CHECK)

CASE II : RANK(A) < RANK$(A^{\#})$

Determine the solution set to

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 1, \\ 2x_1 + 3x_2 + x_3 \phantom{{}- x_4} &= 4, \\ 3x_1 + 5x_2 + 3x_3 - x_4 &= 5. \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{array}\right] \xrightarrow[A_{13}(-3)]{A_{12}(-2)} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & \boxed{1} & 3 & -2 & 2 \\ 0 & 2 & 6 & -4 & 2 \end{array}\right]$$

$$\xrightarrow{A_{23}(-2)} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

$$A^{\#} \xrightarrow{\text{EROs}} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

$$A \xrightarrow{\text{EROs}} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2$, $\text{rank}(A^{\#}) = 3$, $n = 4$

$\text{rank } A < \text{rank } A^{\#} \longrightarrow$ IN CONSISTENT!

$[A \mid b]$

$\longrightarrow Ax = b$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = -2$$

CASE III $\qquad \text{RANK}(A) = \text{RANK}(A^{\#}) < n$

Determine the solution set to

$$\begin{aligned} 5x_1 - 6x_2 + x_3 &= 4, \\ 2x_1 - 3x_2 + x_3 &= 1, \\ 4x_1 - 3x_2 - x_3 &= 5. \end{aligned}$$

$$A^{\#} = \left[\begin{array}{ccc|c} 5 & -6 & 1 & 4 \\ 2 & -3 & 1 & 1 \\ 4 & -3 & -1 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} \boxed{5} & -6 & 1 & 4 \\ 2 & -3 & 1 & 1 \\ 4 & -3 & -1 & 5 \end{array}\right]$$

(could do $A_{31}(-1)$, $A_{21}(-2)$)

P_{ij}
$M_i(k)$
$A_{ij}(k)$

$$\Big\downarrow A_{31}(-1)$$

$$\left[\begin{array}{ccc|c} \boxed{1} & -3 & 2 & -1 \\ 2 & -3 & 1 & 1 \\ 4 & -3 & -1 & 5 \end{array}\right] \xrightarrow[A_{13}(-4)]{A_{12}(-2)} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 3 & -3 & 3 \\ 0 & 9 & -9 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & \boxed{3} & -3 & 3 \\ 0 & 9 & -9 & 9 \end{array}\right] \xrightarrow{M_2(1/3)} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & \boxed{1} & -1 & 1 \\ 0 & 9 & -9 & 9 \end{array}\right]$$

$$\downarrow A_{23}(-9)$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$A^{\#} \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$A \xrightarrow{\text{EROs}} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A^{\#}) = 2$, $\text{rank}(A) = 2$, $n = 3$

$$\begin{array}{l} \rightarrow \\ \rightarrow \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 - 3x_2 + 2x_3 = -1$$

$$x_2 - x_3 = 1$$

DEGREES OF FREEDOM = # OF VARIABLES − # OF $\boxed{\text{CONSTRAINTS}}$

↳ INDEPENDENT

$= 3 - 2 = 1$

$$x_1 - 3x_2 + 2x_3 = -1$$

$$x_2 - x_3 = 1$$

$$x_2 = t$$

$$x_3 = x_2 - 1 = t - 1$$

$$x_1 = -1 + 3x_2 - 2x_3$$

$$= -1 + 3t - 2(t-1) = t + 1$$

$$x_1 = t + 1, \quad x_2 = t, \quad x_3 = t - 1$$

SOLUTION SET $\subseteq \mathbb{R}^3$ $= \left\{ (\overbrace{t+1}^{x_1}, \overbrace{t}^{x_2}, \overbrace{t-1}^{x_3}) : t \in \mathbb{R} \right\}$

(ONE PARAMETER)

Example 2.5.8 Use Gaussian elimination to solve

$$A^{\#} = \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 3 & 1 & 6 & 11 & 16 \\ 2 & 1 & 4 & 4 & 9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + 2x_3 - x_4 &= 3, \\ 3x_1 + x_2 + 6x_3 + 11x_4 &= 16, \\ 2x_1 - x_2 + 4x_3 + 4x_4 &= 9. \end{aligned}$$

Solution: A row-echelon form of the augmented matrix of the system is

$$A^{\#} \xrightarrow{\text{EROs}} \left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$\text{rank } A^{\#} = 2$

$\text{rank } A = 2$

$n = 4$

DEGREES OF FREEDOM $= 4 - 2 = 2$

$(n - r^{\#})$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 - 2x_2 + 2x_3 - x_4 = 3$$

$$x_2 + 2x_4 = 1$$

$x_4 = s$, $x_2 = 1 - 2x_4 = 1 - 2s$

PLUG IN x_2, x_4, x_3 INTO EQ 1. , $x_3 = t$

$$x_1 = 3 + 2x_2 - 2x_3 + x_4$$

$$= 3 + 2(1-2s) - 2t + s$$

$$= 5 - 3s - 2t$$

SOLUTION SET $= \left\{ \left(\underbrace{5-3s-2t}_{x_1}, \underbrace{1-2s}_{x_2}, \underbrace{t}_{x_3}, \underbrace{s}_{x_4} \right) : s,t \in \mathbb{R} \right\}$

$$\boxed{\text{Choose as free variables those variables that } \textbf{do not} \text{ correspond to a leading 1 in a row-echelon form of } A^{\#}.}$$

rank A = rank $A^{\#}$

HOMOGENEOUS EQUATIONS

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0,\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0,
\end{aligned}$$

$Ax = b$

$\downarrow$

0

$Ax = 0$

$x = 0$

$x_1 = 0, x_2 = 0, \ldots, x_n = 0$

Corollary 2.5.10 The homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is consistent for any coefficient matrix A, with a solution given by $\mathbf{x} = \mathbf{0}$.

Corollary 2.5.11 A homogeneous system of m linear equations in n unknowns, with $m < n$, has an infinite number of solutions.

$$\Rightarrow \underset{\substack{\parallel \\ \text{rank } A^{\#}}}{\text{rank } A} \leq m < n \longrightarrow \text{INFINITE SOLN.}$$

BREAK
TILL
10 : 10 AM

SKIP

Determine the solution set to $A\mathbf{x} = \mathbf{0}$, if $A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 3 & 7 \end{bmatrix}$.

SAMPLE MIDTERM 1, Q5

rank A

$$A = \begin{bmatrix} \boxed{1} & -2 & 1 & 3 \\ 3 & -6 & 2 & 7 \\ 4 & -8 & 3 & 10 \end{bmatrix}$$

$$\downarrow A_{12}(-3),\ A_{13}(-4)$$

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & \boxed{-1} & -2 \\ 0 & 0 & -1 & -2 \end{bmatrix} \xrightarrow{M_2(-1)} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$A_{23}(1)$

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{RANK}(\underset{\sim}{A}) = 2$

SAMPLE MIDTERM 1
Q6

GAUSS – JORDAN

$$x_1 + 2x_2 + x_3 = 1$$

$$3x_1 + 5x_2 + x_3 = 3$$

$$2x_1 + 6x_2 + 7x_3 = 1$$

AUGMENTED MATRIX $= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{array}\right]$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{array}\right] \xrightarrow[A_{13}(-2)]{A_{12}(-3)} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & \boxed{-1} & -2 & 0 \\ 0 & 2 & 5 & -1 \end{array}\right] \xrightarrow{M_2(-1)} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 2 & 5 & -1 \end{array}\right]$$

$$\xrightarrow[A_{21}(-2)]{A_{23}(-2)} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$$

$$\Big\downarrow \; A_{31}(3),\; A_{32}(-2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$x_1 = -2$$

$$x_2 = 2$$

$$x_3 = -1$$

$$\boxed{(x_1, x_2, x_3) = (-2, 2, -1)}$$

ANSWER!