

Math 165: Linear Algebra w/ Diff. Equations

Sample Midterm 1

~~Jan 21, 2022~~

July 7th, 2022

NAME (please print legibly): _____

Your University ID Number: _____

- The exam will be 75 minutes long. You will get extra time in the end to upload the exam to Gradescope.
- There are 7 pages.
- A formula sheet is provided.
- No calculators, phones, electronic devices, books, notes are allowed during the exam. The only materials you are allowed to use are pen/pencil and paper. In particular, you are NOT allowed to take the exam on a tablet.
- You are allowed to use a phone or tablet to take photographs of your answer sheet once the exam is over. If you finish early, you must take permission before taking photographs. Once you start taking photographs, you are not allowed to write.
- **Show all work and justify all answers as much as possible.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

QUESTION	VALUE	SCORE
1	0	
2	20	
3	15	
4	20	
5	25	
6	20	
TOTAL	100	

1. (0 points) Copy the following honesty pledge on to your answer sheet. Remember to sign and date it.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: _____

2. (20 points)

Consider the differential equation

$$y' = -2xy^2.$$

(a) What are the isoclines for this differential equation?

 $k \rightarrow \text{FIXED}$ ISOCLINE: $f(x, y) = k$

$$y' = f(x, y)$$

$$\boxed{-2xy^2 = k}$$

(b) Are there any equilibrium solutions? Identify them.

$$-2xy^2 = 0 \Rightarrow x = 0 \text{ OR } y = 0$$

$$\text{YES} \rightarrow \boxed{y = 0}, y' = 0$$

(c) Solve the initial value problem given by this differential equation together with the condition $y(0) = 1$.

$$y' = -2xy^2, \quad y(0) = 1$$

METHOD 1:

$$-\frac{dy}{y^2} = 2x dx$$

(SEPARABLE)
 $\because y' = \frac{dy}{dx}$

$$\Rightarrow \int \frac{-dy}{y^2} = \int 2x dx + C$$

$$\Rightarrow \frac{1}{y} = x^2 + C \Rightarrow y = \frac{1}{x^2 + C}$$

$$\int 2x dx = x^2$$

$$\int -y^{-2} dy = \frac{1}{y}$$

$$1 = y(0) = \frac{1}{C} \Rightarrow C = 1 \Rightarrow \boxed{y(x) = \frac{1}{x^2 + 1}}$$

METHOD 2:

$$\int_1^y (-y^{-2}) dy = \int_0^x 2x dx \Rightarrow \frac{1}{y} - 1 = x^2$$

$$\Rightarrow y(x) = \frac{1}{x^2 + 1}$$

3. (15 points) Find the general solution for the following differential equation:

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{(1+x^2)^2}$$

FIRST ORDER
LINEAR ODE

$$P(x) = \frac{2x}{1+x^2}$$

$$\Rightarrow I(x) = e^{\int P(x) dx} = e^{\ln(1+x^2)} = 1+x^2$$

$$\int P(x) dx = \int \left(\frac{2x}{1+x^2} \right) dx$$

$$u = 1+x^2, \quad du = 2x dx$$

$$\int \frac{du}{u} = \ln u = \ln(1+x^2)$$

MULTIPLY BY $I(x)$

$$\frac{d}{dx} \left(\frac{I(x)y}{I(x)} \right) = (1+x^2) \frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \left((1+x^2)y \right) = \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)y = \int \frac{dx}{1+x^2} + C \rightarrow \text{Arctan } x$$

$$(1+x^2)y = \text{Arctan } x + C$$

$$\Rightarrow y = \frac{\text{Arctan } x + C}{1+x^2}$$

ONE
PARAMETER
FAMILY

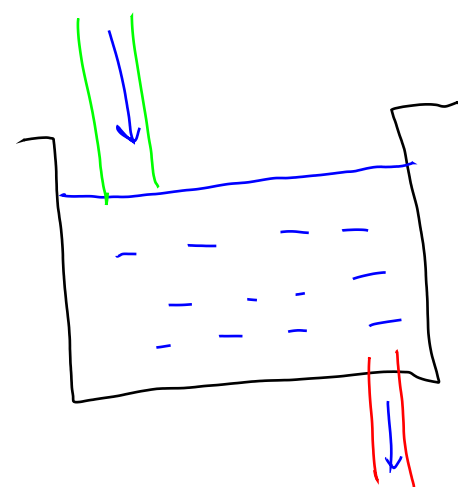
4. (20 points) A tank contains $8L$ of water in which is dissolved $32g$ of chemical. A solution containing $2g/L$ of the chemical flows into the tank at a rate of $4L$ per minute and the well-stirred mixture flows out at a rate of $2L$ per minute.

$V(t) \rightarrow$ VOL AT TIME t
 $A(t) \rightarrow$ AMOUNT AT TIME t
 (with scribbles and corrections)

$A(20)$ ← (a) Determine the amount of chemical in the tank after 20 minutes.

$\frac{A(20)}{V(20)}$ ← (b) What is the concentration of chemical in the tank at that time?

$C_1 = 2$
 $r_1 = 4$



$V(0) = 8$
 $A(0) = 32$

$r_2 = 2L$
 $C_2(t) = \frac{A(t)}{V(t)}$

$\frac{dV}{dt} = r_1 - r_2 = 4 - 2 = 2$

$\Rightarrow \int_8^{V(t)} dV = \int_0^t 2 dt \Rightarrow V(t) - 8 = 2t \Rightarrow V(t) = 2t + 8$

$\frac{dA}{dt} = r_1 C_1 - r_2 C_2 = (2)(4) - (2) \frac{A}{V} = 8 - \frac{2A}{2t+8}$

$\frac{dA}{dt} + \frac{A}{t+4} = 8$

FIRST ORDER LINEAR ODE \Rightarrow INTEGRATING FACTOR

$\frac{d}{dt} ((t+4)A(t)) = 8t + 32$

$\int_0^{20} \frac{d}{dt} (t+4)A(t) dt = \int_0^{20} (8t+32) dt$

5. (25 points) Compute the rank of the following matrix:

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 3 & -6 & 2 & 7 \\ 4 & -8 & 3 & 10 \end{bmatrix}$$

(SEE LECTURE 7
NOTES)

6. (20 points) Use Gauss-Jordan elimination to solve the following system of equations:

$$x_1 + 2x_2 + x_3 = 1,$$

$$3x_1 + 5x_2 + x_3 = 3,$$

$$2x_1 + 6x_2 + 7x_3 = 1.$$

(SEE LECTURE 7
NOTES)