MATH 165 (SUMMER '22, SESH B2)

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OFF HRS:
T – 9:00 PM – 10:00 PM (ET)
F – 3:00 PM – 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEB PAGE $\boxed{\text{https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html}}$

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1-7 ARE UPLOADED.

2. WW 03 - WAS DUE SATURDAY (9th JULY) AT 11:00 PM ET
EXT? <- WW 04 - IS DUE TUESDAY (12th JULY) AT 11:00 PM ET
WW 05 - IS DUE SATURDAY (16th JULY) AT 11:00 PM ET

3. MIDTERM 1 TODAY !

4. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE !

§ 2.6 INVERSE OF A SQUARE MATRIX

$A \to n \times n$ MATRIX.

(Q: DOES THERE EXIST B:)

$$AB = I_n \quad , \quad BA = I_n$$

MOTIVATION: $A\vec{x} = \vec{b}$

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n \end{cases}$$

SUPPOSE, $\exists B$, s.t. $\boxed{BA = I_n}$

$A\vec{x} = \vec{b} \implies B(A\vec{x}) = B\vec{b}$

(PREMULTIPLY BY B)

$$\| $$

$(BA)\vec{x} = (I_n)\vec{x}$

$= I_n\vec{x} = \vec{x}$

$$I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \\ & & & & 1 \end{bmatrix}$$

$\vec{x} = B\vec{b}$

$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}$

$\left.\begin{matrix} B \to A \\ \vec{b} \end{matrix}\right\}$ FIXED

$m \times n$

$$A I_n = I_m A = A$$

Theorem 2.6.1 Let A be an $n \times n$ matrix. Suppose B and C are both $n \times n$ matrices satisfying

$$AB = BA = I_n \tag{2.6.4}$$

$$AC = CA = I_n \tag{2.6.5}$$

respectively. Then $B = C$.

Pf

$$B = BI_n = B(AC)$$

$$= (BA)C = I_n C = C$$

$$\Rightarrow B = C \quad \blacksquare$$

DEFINITION 2.6.2

Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix A^{-1} satisfying

$$AA^{-1} = A^{-1}A = I_n,$$

then we call A^{-1} *the* matrix **inverse** to A, or just *the* inverse of A. We say that A is **invertible** if A^{-1} exists.

Invertible matrices are sometimes called **nonsingular**, while matrices that are not invertible are sometimes called **singular**.

INVERTIBLE

Example 2.6.3 If $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}$, verify that $B = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ is the inverse of A.

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad BA = AB = I,$

$$B = A^{-1}.$$

Theorem 2.6.5 If A^{-1} exists, then the $n \times n$ system of linear equations

$$A\mathbf{x} = \mathbf{b}$$

has the *unique* solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

for every $\mathbf{b}$ in $\mathbb{R}^n$.

Pf

$$\Rightarrow A^{-1}(Ax) = A^{-1}b$$

$$\Rightarrow x = Ix = (A^{-1}A)x = A^{-1}(Ax) = A^{-1}b$$

$$\Rightarrow \text{Rank } A = \text{Rank } A^{\#} = n$$

$$A^{\#} = [A \mid b]$$

Theorem 2.6.6 An $n \times n$ matrix A is invertible if and only if $\operatorname{rank}(A) = n$.

Pf ($\Rightarrow$) A IS INVERTIBLE

$\Rightarrow$ $A\vec{x} = \vec{b}$ HAS A UNIQUE SOLN.

$\Rightarrow$ $\operatorname{rank} A = \operatorname{rank} [A \mid b] = n$

($\Leftarrow$) $\operatorname{rank} A = n$

$AX = I$

$$I = \begin{array}{c} \begin{array}{cccc} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{array} \\ \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \end{array}$$

$$I = \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix}$$

$$\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow \text{jth entry}$$

$$X = \begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix}$$

$$AX = A \begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix} = \begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix}$$

$A\vec{x}_j = \vec{e}_j \rightarrow \vec{x}_j$ EXISTS & IS UNIQUE

$\boxed{\text{rank } A = \text{rank } [A \mid \vec{e}_j] = n}$

$A^{-1} = [x_1 \; \ldots\ldots \; x_n]$

$\Rightarrow$ A IS INVERTIBLE

GAUSS-JORDAN METHOD FOR INVERSES

$$AX = I_n$$

$$X = \begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix}$$

$$I_n = \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix}$$

$$\boxed{A\vec{x}_j = \vec{e}_j}$$

$$e_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ jth entry}$$

$$A\vec{x} = \vec{b}$$

$$\left[\begin{array}{c|c} A & b \end{array}\right] \xrightarrow{\text{ERO}} \text{ROW ECHELON FORM (GAUSSIAN)}$$

$$\left[\begin{array}{c|c} A & b \end{array}\right] \xrightarrow{\text{ERO}} \text{REDUCED ROW ECHELON FORM (GAUSS-JORDAN)}$$

$$[A \mid b] \xrightarrow[\text{GAUSS-JORDAN}]{\text{ERO}} \left[\begin{array}{cccc|c} 1 & & & 0 & c_1 \\ & 1 & & & \vdots \\ & & 1 & & \vdots \\ 0 & & & 1 & c_n \end{array}\right]$$

$A \to n \times n$ MATRIX INVERTIBLE

$A\vec{x} = \vec{b}$, $A \to$ INVERTIBLE $(n \times n)$

$$[A \mid \vec{b}] \xrightarrow{\text{EROs}} [I_n \mid \vec{x}]$$

$(A\vec{x}' = \vec{b}')$

$$[A \mid \vec{b}'] \xrightarrow[\text{EROs}]{\text{SAME}} [I_n \mid \vec{x}']$$

$$A\vec{x}_j = \vec{e}_j$$

$$\left[\begin{array}{c|c} A & \vec{e}_j \end{array}\right] \xrightarrow[\text{EROs}]{\text{SAME}} \left[\begin{array}{c|c} I_n & \vec{x}_j \end{array}\right]$$

$$\left[\begin{array}{c|c} A & \underbrace{\begin{array}{c|c|c|c} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{array}}_{I_n} \end{array}\right] \xrightarrow[\text{EROs}]{\text{SAME}} \left[\begin{array}{c|c} I_n & \underbrace{\begin{array}{c|c|c|c} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array}}_{X} \end{array}\right]$$

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{c|c} I_n & X \end{array}\right]$$

ALGO: A^{-1},

$$\left[\begin{array}{c|c} A & I_n \end{array}\right]$$

$\Big\downarrow$ EROs FOR REDUCED ROW ECHELON FORM

$$\left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$$

Example 2.6.8 Find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix}$.

$$[\,A \mid I\,]$$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 5 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{A_{13}(-3)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 2 & -10 & -3 & 0 & 1 \end{array}\right] \xrightarrow[A_{23}(-2)]{A_{21}(-1)}$$

$$\xrightarrow{A_{21}(-1),\ A_{23}(-2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -14 & -3 & -2 & 1 \end{array}\right]$$

$$\Big\downarrow M_3(-1/14)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1\checkmark & 1 & -1 & 0 \\ 0 & 1 & 2\checkmark & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 3/14 & 1/7 & -1/14 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3/14 & 1/7 & -1/14 \end{array}\right]$$

$$\Big\downarrow \; \begin{array}{l} A_{32}(-2) \\ A_{31}(-1) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11/14 & -8/7 & 1/14 \\ 0 & 1 & 0 & -3/7 & 5/7 & 1/7 \\ 0 & 0 & 1 & 3/14 & 1/7 & -1/14 \end{array}\right]$$

$\to A^{-1}$

Example 2.6.9 Continuing the previous example, use A^{-1} to solve the system

$$\begin{aligned} x_1 + \quad x_2 + 3x_3 &= 2, \\ x_2 + 2x_3 &= 1, \\ 3x_1 + 5x_2 - \quad x_3 &= 4. \end{aligned}$$

$$A\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{x} = A^{-1} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 11/14 & -8/7 & 1/14 \\ -3/7 & 5/7 & 1/7 \\ 3/14 & 1/7 & -1/14 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

Theorem 2.6.10 Let A and B be invertible $n \times n$ matrices. Then

1. A^{-1} is invertible and $(A^{-1})^{-1} = A$.
2. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.