MATH 165 (SUMMER '22, SESH B2)

ANURAG SAHAY
OFF HRS: BY APPT.
email: anuragsahay@rochester.edu

TA: PABLO BHOWMIK
OFF HRS:
T — 9:00 PM – 10:00 PM (ET)
F — 3:00 PM – 4:00 PM (ET)

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEB PAGE: https://people.math.rochester.edu/grads/asahay/summer2022/math165/index.html

SHORT URL: bit.ly/sahay165

NOTE: ALL IMAGES ARE FROM THE (GOODE & ANNIN 4TH EDITION)

ANNOUNCEMENTS / NOTES

1. MATERIALS FOR LECTURES 1-8 ARE UPLOADED.

2. WW 03 - WAS DUE SATURDAY (9th JULY) AT 11:00 PM ET
   WW 04 - IS DUE ~~TUESDAY (12th JULY)~~ WED (13th JULY) AT 11:00 PM ET
   WW 05 - IS DUE SATURDAY (16th JULY) AT 11:00 PM ET

3. MIDTERM 1 WILL BE GRADED BY TONIGHT.

4. REMINDER : PLEASE KEEP VIDEOS ON, IF POSSIBLE!

§ 2.6 INVERSE OF A SQUARE MATRIX

RECALL

DEFINITION 2.6.2

Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix A^{-1} satisfying

$$AA^{-1} = A^{-1}A = I_n,$$

then we call A^{-1} *the* matrix **inverse** to A, or just *the* inverse of A. We say that A is **invertible** if A^{-1} exists.

Invertible matrices are sometimes called **nonsingular**, while matrices that are not invertible are sometimes called **singular**.

RECALL

Theorem 2.6.6 An $n \times n$ matrix A is invertible if and only if $\operatorname{rank}(A) = n$.

$$(AB)^{-1} \neq A^{-1}B^{-1}$$

Theorem 2.6.10 Let A and B be invertible $n \times n$ matrices. Then

1. A^{-1} is invertible and $(A^{-1})^{-1} = A$.
2. AB is invertible and $\underline{(AB)^{-1} = B^{-1}A^{-1}}$.
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

$(AB)C = I$

$B = A^{-1} \iff AB = BA = I$

$\iff A = B^{-1} = (A^{-1})^{-1}$

$C = B^{-1}A^{-1}$

$(AB)C = (AB)(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1}$

$= AIA^{-1}$

$= AA^{-1} = I$

$$\left(AB = BA = I \right)^T \qquad ((AB)^T = B^T A^T)$$

$$B^T A^T = A^T B^T = I^T = I$$

$$\Rightarrow \quad B^T = (A^T)^{-1}$$

§3.1 (THE DEFN OF) THE DETERMINANT

Q. HOW TO CHECK IF A IS INVERTIBLE?

ANS? $\rightsquigarrow$ IS $\underbrace{\text{Rank}(A) = n}_{???}$?

CASE I: $A \to 1 \times 1$ $\qquad A = [a_{11}]$

A IS INVERIBLE $\iff \text{rk} A = 1 \iff a_{11} \neq 0$

CASE II : $A \to 2 \times 2$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{M_1(1/a_{11})} \begin{bmatrix} 1 & a_{12}/a_{11} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Big\downarrow A_{12}(-a_{21})$$

$$\begin{bmatrix} 1 & a_{12}/a_{11} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} \end{bmatrix}$$

$$a_{22} - \frac{a_{12}a_{21}}{a_{11}} \neq 0$$

A IS INVERTIBLE

$\Updownarrow$

$rk\, A = 2$

$\Updownarrow$

$a_{11}a_{22} - a_{12}a_{21} \neq 0$

CASE III : $A \rightarrow 3 \times 3$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A IS INVERTIBLE $\Leftrightarrow$ $Rk(A) = 3$

$$\Updownarrow$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \neq 0$$

GENERAL CASE ?

I : $a_{11} \neq 0$

II : $a_{11}a_{22} - a_{12}a_{21} \neq 0$

III : $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \neq 0$

ANS : PERMUTATIONS $\left(a_{1p_1}a_{2p_2}a_{3p_3}\cdots a_{np_n}\right)$

PERMUTATIONS

$123 \ldots n \longrightarrow p_1 \cdots p_n$

$n = 3$, 321, 132, 123

$122 \rightarrow$ NOT A PERMUTATION

Example 3.1.1 There are precisely six distinct permutations of the integers 1,2, and 3:

$$(1, 2, 3), \quad (1, 3, 2), \quad (2, 1, 3), \quad (2, 3, 1), \quad (3, 1, 2), \quad (3, 2, 1).$$

$$\Big(\underbrace{\underline{a}}_{\text{3 OPTIONS}},\ \overbrace{\underline{b}}^{\text{2 OPTIONS}},\ \underbrace{\underline{c}}_{\text{1 OPTIONS}} \Big)$$

$\# = 3 \times 2 \times 1 = 6$

Theorem 3.1.2 There are precisely $n!$ distinct permutations of the integers $1, 2, \ldots, n$.

$$\Big(\underset{n}{\underline{\quad}},\ \underset{n-1}{\underline{\quad}},\ \underset{n-2}{\underline{\quad}},\ \ldots,\ \underset{1}{\underline{\quad}} \Big)$$

$\#$ OF CHOICES $= n(n-1)\cdots(1)$

$= n!$

INVERSION

A PAIR (i, j) s.t $i < j$ BUT $p_i > p_j$
IS CALLED AN INVERSION

\# OF INVERSIONS $= N(p_1, p_2 \dots p_n)$

Example 3.1.3 Find the number of inversions in the permutations $\boxed{(1, 3, 2, 4, 5)}$ and $\boxed{(2, 4, 5, 3, 1)}$.

→ 6 INVERSIONS

(2,1) (5,1)
(4,1) (5,3)
(4,3) (3,1)

1 INVERSION
(3,2)

$$(1\,3\,2\,4) = (p_1\,p_2\,p_3\,p_4)$$

$(1,2) \longrightarrow$	$p_1 = 1$	$< p_2 = 3$	NOT
$(1,3) \longrightarrow$	$p_1 = 1$	$< \ldots$	NOT
$(1,4) \longrightarrow$	$p_1 = 1$	$< \ldots$	NOT
$(2,3) \longrightarrow$	$p_2 = 3$	$> p_3 = 2$	INV.
$(2,4) \longrightarrow$	$p_2 = 3$	$< p_4 = 4$	NOT
$(3,4) \longrightarrow$	$p_3 = 2$	$< p_4 = 4$	NOT

DEFINITION 3.1.4

1. If $N(p_1, p_2, \ldots, p_n)$ is an even integer (or zero), we say $(p_1, p_2, \ldots, p_n)$ is an **even permutation**. We also say that $(p_1, p_2, \ldots, p_n)$ has **even parity**.
2. If $N(p_1, p_2, \ldots, p_n)$ is an odd integer, we say $(p_1, p_2, \ldots, p_n)$ is an **odd permutation**. We also say that $(p_1, p_2, \ldots, p_n)$ has **odd parity**.

e.g. $(1\ 3\ 2\ 4\ 5) \to$ ODD

$(2\ 4\ 5\ 3\ 1) \to$ EVEN

DEFN

$A \to n \times n$

DEFINITION 3.1.8

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **determinant of** A, denoted $\det(A)$, is defined as follows:

$$\det(A) = \sum \sigma(p_1, p_2, \cdots, p_n) a_{1p_1} a_{2p_2} a_{3p_3} \cdots a_{np_n}, \tag{3.1.3}$$

OVER ALL PERMUTATIONS OF $(1, \cdots, n)$

where the summation is over the $n!$ distinct permutations $(p_1, p_2, \ldots, p_n)$ of the integers $1, 2, 3, \ldots, n$. The determinant of an $n \times n$ matrix is said to have **order** n.

We sometimes denote $\det(A)$ by

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}.$$

$$\sigma(p_1, \cdots, p_n) = \begin{cases} +1 & \text{EVEN} \\ -1 & \text{ODD} \end{cases}$$

$(-1)^{N(p_1, \cdots, p_n)}$

$a_{11}a_{22} - a_{12}a_{21}$

$= (-4)(5) - (-2)(6)$

$= -8$

-6

Example 3.1.10 Evaluate

(a) $|-6|$.

(b) $\begin{vmatrix} -4 & 6 \\ -2 & 5 \end{vmatrix}$.

SKIP

(c) $\cancel{\begin{vmatrix} 2 & -5 & 2 \\ 6 & 1 & 0 \\ -3 & -1 & 4 \end{vmatrix}}$.

(d) $\begin{vmatrix} 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & 6 \\ -2 & 1 & 0 & 0 \\ -7 & 3 & 0 & 0 \end{vmatrix}$.

$a_{jp_j} = 0$

$\nearrow$ $j \in \{1,2\},\ p_j \in \{1,2\}$

$\searrow$ $j \in \{3,4\},\ p_j \in \{3,4\}$

$j \in \{1,2\}$, $p_j \in \{1,2\}$

$j \in \{3,4\}$, $p_j \in \{3,4\}$

ADMSISSIBLE :

$(3\ 4\ 1\ 2) \rightarrow$ EVEN

$(4\ 3\ 1\ 2) \rightarrow$ ODD

$(3\ 4\ 2\ 1) \rightarrow$ ODD

$(4\ 3\ 2\ 1) \rightarrow$ EVEN

$$\text{DET} = a_{13}\, a_{24}\, a_{31}\, a_{42} - a_{14}\, a_{23}\, a_{31}\, a_{42} - a_{13}\, a_{24}\, a_{32}\, a_{41} + a_{14}\, a_{23}\, a_{32}\, a_{41}$$

FORGET THE DEFINITION!

LEARN TO COMPUTE.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det A = a_{11} a_{22} - a_{12} a_{21}$$

DETOUR:

§ 3.3 COFACTOR EXPANSION

DEFINITION 3.3.1

Let A be an $n \times n$ matrix. The **minor**, M_{ij}, of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row vector and jth column vector of A.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & \boxed{5} \\ -1 & 0 & 2 \end{bmatrix}$$

$$M_{23} = \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}$$

$$= (1)(0) - (-1)(3)$$

$$= 3$$

Determine the minors M_{11}, M_{21}, and M_{32} for

$$A = \begin{bmatrix} -4 & 9 & 1 \\ -2 & -2 & 5 \\ 6 & 3 & 1 \end{bmatrix}.$$

$$M_{11} = \begin{vmatrix} -2 & 5 \\ 3 & 1 \end{vmatrix} = (-2)(1) - (3)(5) = -17$$

$$\left(C_{11} = (-1)^{1+1} M_{11} = -17 \right)$$

$$M_{21} = \begin{vmatrix} 9 & 1 \\ 3 & 1 \end{vmatrix} = (9)(1) - (3)(1) = 6$$

$$\left(C_{21} = (-1)^{2+1} M_{21} = -6 \right)$$

$$M_{32} = \begin{vmatrix} -4 & 1 \\ -2 & 5 \end{vmatrix} = (-4)(5) - (-2)(1) = -18$$

$$\left[C_{32} = (-1)^{3+2} M_{32} = 18 \right]$$

DEFINITION 3.3.4

Let A be an $n \times n$ matrix. The **cofactor**, C_{ij}, of the element a_{ij} is defined by

$$C_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the minor of a_{ij}.

$$\begin{vmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{vmatrix}.$$

Theorem 3.3.8 **(Cofactor Expansion Theorem)**

Let A be an $n \times n$ matrix. If we multiply the elements in any row (or column) of A by their cofactors, then the sum of the resulting products is $\det(A)$. Thus,

1. If we expand along row i,

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik}.$$

2. If we expand along column j,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{k=1}^{n} a_{kj}C_{kj}.$$

COFACTOR EXPANSION

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

Example 3.3.9 Use the Cofactor Expansion Theorem along (a) row 1, ~~(b) column~~ 2 to find $\begin{vmatrix} -8 & 7 & 0 \\ -1 & -3 & 6 \\ 4 & -2 & -2 \end{vmatrix}$.

a_{11}

$$M_{11} = \begin{vmatrix} -3 & 6 \\ -2 & -2 \end{vmatrix}$$

$$= 18$$

$$C_{11} = +M_{11} = 18$$

a_{12}

$$M_{12} = \begin{vmatrix} -1 & 6 \\ 4 & -2 \end{vmatrix}$$

$$= -22$$

$$C_{12} = -M_{12}$$

$$= 22$$

a_{13}

$$M_{13} = ?$$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= (-8)(18) + (7)(22) + 0$$

$$=$$

$$\text{Evaluate} \begin{array}{c} \begin{array}{cccc} + & - & + & \\ \end{array} \\ \begin{vmatrix} -6 & 2 & 1 & 5 \\ 7 & -3 & 0 & 2 \\ 9 & -4 & 0 & 8 \\ 1 & 0 & -2 & 0 \end{vmatrix} \end{array}.$$

31

~~32~~

~~33~~

34

EXPAND ALONG COLUMN

$$= (1)(+1)\begin{vmatrix} 7 & -3 & 2 \\ 9 & -4 & 8 \\ 1 & 0 & 0 \end{vmatrix}$$

$$+ (-2)(-1)\begin{vmatrix} -6 & 2 & 5 \\ 7 & -3 & 2 \\ 9 & -4 & 8 \end{vmatrix}$$

BREAK TILL

10 : 20 AM

$\S$ 3.2 PROP. OF DETERMINANTS

WHAT DOES ONE DO IF n IS LARGE?

SIMPLIFY!

Theorem 3.2.1 If A is an $n \times n$ upper or lower triangular matrix, then

$$\det(A) = a_{11}a_{22}a_{33}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

$A \to$ UPPER TRIANGULAR

$$\det A = \prod_{j=1}^{n} a_{jj} = a_{11} a_{22} \cdots a_{nn}$$

$$\begin{vmatrix} a_{11} & & & * \\ & a_{22} & & \\ & 0 & \ddots & \\ & & & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & & & * \\ & a_{33} & & \\ & 0 & \ddots & \\ & & & a_{nn} \end{vmatrix} = a_{11} a_{22} \begin{vmatrix} \cdots \end{vmatrix}$$

$$= a_{11} \cdots a_{nn}$$

$$A = \begin{bmatrix} -6 & 4 & 9 & -2 \\ 0 & 2 & 3 & 8 \\ 0 & 0 & -5 & 6 \\ 0 & 0 & 0 & -3 \end{bmatrix},$$

$$\det = (-6)(2)(-5)(-3)$$

$$= -180$$

EROs & DET.

P_{ij}

$M_j(k)$

$A_{ij}(k)$

P_{ij}

P1. If B is the matrix obtained by permuting two rows of A, then

$$\det(B) = -\det(A).$$

$$\begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} \stackrel{?}{=} - \begin{vmatrix} 1 & 6 \\ 3 & 5 \end{vmatrix}$$

$\shortparallel$ $\qquad\qquad$ $\shortparallel$

$3\times 6 - 1\times 5 = 13$ $\qquad\qquad$ $1\times 5 - 3\times 6 = -13$

$M_j(k)$

$k = 0$ IS ALSO OKAY

P2. If B is the matrix obtained by multiplying one row of A by any[2] scalar k, then

$$\det(B) = k\det(A).$$

$$A = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} \qquad \det A = -\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$M_3(2)$

$$\det B = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 2 & 0 \end{vmatrix} = -\begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} = 2 = 2\det A$$

P4. For any scalar k and $n \times n$ matrix A, we have

$$\det(kA) = k^n \det(A).$$

$(P2 \Rightarrow P4)$

$A_{ij}(k)$

P3. If B is the matrix obtained by adding a multiple of any row of A to a different row of A, then

$$\det(B) = \det(A).$$

det = 7

MULT $(g,h,i) \to 4$

PERM (g,h,i) to top

Suppose that $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is some 3×3 matrix with $\det(A) = 7$. Compute

1. ADD (a,b,c) & (d,e,f)

$A_{21}(1)$

$$\det \begin{bmatrix} 4g & 4h & 4i \\ a+d & b+e & c+f \\ a-2g & b-2h & c-2i \end{bmatrix}.$$

2. ADD $-2(g,h,i)$ to (a,b,c)

$$7 = \begin{vmatrix} a & b & c \\ a+d & b+e & c+f \\ g & h & i \end{vmatrix} \xrightarrow{A_{31}(-2)} \begin{vmatrix} a-2g & b-2h & c-2i \\ a+d & b+e & c+f \\ g & h & i \end{vmatrix} = 7$$

$$\begin{vmatrix} a-2g & b-2h & c-2i \\ a+d & b+e & c+f \\ g & h & i \end{vmatrix} \xrightarrow{P_{13}} \begin{vmatrix} g & h & i \\ a+d & b+e & c+f \\ a-2g & b-2h & c-2i \end{vmatrix} = -7$$

$$\Big\downarrow M_1(4)$$

$$-28 = (4)(-7) = \begin{vmatrix} 4g & 4h & 4i \\ a+d & b+e & c+f \\ a-2g & b-2h & c-2i \end{vmatrix}$$

$A_{ij}(k)$, P_{ij}, $M_j(k)$

Evaluate $\begin{vmatrix} 3 & 4 & -1 & 5 \\ 1 & 2 & -1 & 3 \\ -2 & -2 & 2 & -7 \\ -4 & -3 & -2 & -8 \end{vmatrix}$.

$$\xrightarrow{P_{12}} -\begin{vmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & -1 & 5 \\ -2 & -2 & 2 & -7 \\ -4 & -3 & -2 & -8 \end{vmatrix} \xrightarrow{A_{12}(-3),\ A_{13}(2),\ A_{14}(4)} -\begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 2 & -4 \\ 0 & 2 & 0 & -1 \\ 0 & 5 & -6 & 4 \end{vmatrix}$$

$$\xrightarrow{M_2(-1/2)} 2\begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 5 & -6 & 4 \end{vmatrix} \xrightarrow{A_{23}(-2),\ A_{24}(-5)} 2\begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & -1 & -6 \end{vmatrix}$$

$$2\begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & -1 & -6 \end{vmatrix} \xrightarrow{P_{34}} -2\begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & \boxed{2} & -5 \end{vmatrix}$$

$$\xrightarrow{A_{34}(2)} -2\begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & -17 \end{vmatrix} = -2\,(1)(1)(-1)(-17)$$

$$= -34$$

Theorem 3.2.5 Let A be an $n \times n$ matrix. The following conditions on A are equivalent.

(a) A is invertible.

(g) $\det(A) \neq 0$.

Corollary 3.2.6 The homogeneous $n \times n$ linear system $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions if and only if $\det(A) = 0$, and has only the trivial solution if and only if $\det(A) \neq 0$.

Example 3.2.7 Verify that the matrix $A = \begin{bmatrix} 3 & -1 & 2 \\ 7 & 0 & 1 \\ -2 & 3 & 9 \end{bmatrix}$ is invertible. What can be concluded about the solution to $A\mathbf{x} = \mathbf{0}$?

$$\det A = (-1)(-1)\begin{vmatrix} 7 & 1 \\ -2 & 9 \end{vmatrix}$$

$$+ 3(-1)\begin{vmatrix} 3 & 2 \\ 7 & 1 \end{vmatrix}$$

$$= 65 + 33 = 98 \neq 0$$

$$Ax = 0 \Rightarrow x = A^{-1}(0) = 0$$

Example 3.2.8 Verify that the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -3 & 0 & -3 \end{bmatrix}$ is not invertible and determine a set of real solutions to the system $A\mathbf{x} = \mathbf{0}$.

$$\det A = \begin{vmatrix} 1 & 1 \\ -3 & -3 \end{vmatrix} = (1)(-3) - (1)(-3) = 0$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & -3 & 0 \end{array}\right] \xrightarrow{A_{13}(3)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 = 0$

$x_3 = 0$

$x_2 = t \Rightarrow x_1 = -t$

$S = \{(-t, 0, t) : t \in \mathbb{R}\}$

P5. $\det(A^T) = \det(A)$.

DETERMINANTS $\longrightarrow$ ROWS & COLUMNS ARE EQUIVALENT

P7. If A has a row (or column) of zeros, then $\det(A) = 0$.

P8. If two rows (or columns) of A are scalar multiples of one another, then $\det(A) = 0$.

$$\begin{array}{ccc} & j & \\ \left| \begin{array}{c} \\ \\ * \\ \\ \\ \end{array} \right. & \boxed{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}} & \left. \begin{array}{c} \\ \\ * \\ \\ \\ \end{array} \right| \end{array} \longrightarrow$$

COFACTOR EXPANSION

$$\sum_{i=1}^{n} a_{ij} C_{ij} = 0$$

$$\xrightarrow{CM_j(0)}$$

$$\det(A') = k \det(A)$$

$$A \xrightarrow{M_j(k)} A' \longrightarrow \det A = 0 \det(A) = 0$$

P7. If A has a row (or column) of zeros, then $\det(A) = 0$.

P8. If two rows (or columns) of A are scalar multiples of one another, then $\det(A) = 0$.

$$\begin{array}{c} \\ i \to \\ \\ j \to \\ \\ \end{array} \begin{vmatrix} & * & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & * & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & * & & \end{vmatrix} \xrightarrow{A_{ij}(-\lambda)} \text{CREATE A ROW OF ZEROES}$$

$$a_{jk} = \lambda a_{ik}$$