

MATH 165 (SUMMER '22, SESS# B2)

MIDTERM 1

SOLUTIONS

(MORNING SLOT)

2. (20 points)

Consider the differential equation

$$y' = -3x^2(y - 1)^2.$$

(a) What are the isoclines for this differential equation?

LET $k \in \mathbb{R}$ BE FIXED AND ARBITRARY.

$$y' = f(x, y) = -3x^2(y - 1)^2$$

ISOCINES: $f(x, y) = k \rightarrow$

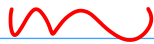
$$\boxed{-3x^2(y - 1)^2 = k}$$

(b) Are there any equilibrium solutions? Identify them.

AN EQ. SOLN IS $y = c$ s.t. $f(x, y) \equiv 0$.

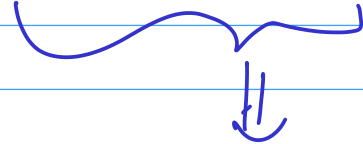
$$f(x, y) = -3x^2(y-1)^2 = 0$$

$$\Rightarrow x = 0 \quad \text{OR} \quad y - 1 = 0$$



NOT
AN EQ.
SOLN.

(\because NOT OF
THE FORM
 $y = c$)



$$y = 1$$

YES!

(c) Solve the initial value problem given by this differential equation together with the condition $y(0) = 2$.

$$y' = -3x^2(y-1)^2$$

SEPARABLE! \rightsquigarrow $\frac{-dy}{(y-1)^2} = 3x^2 dx$

$$\Rightarrow -\int \frac{dy}{(y-1)^2} = \int 3x^2 dx + C$$

$$\Rightarrow \frac{1}{y-1} = x^3 + C$$

$$y(0) = 2 \Rightarrow \frac{1}{2-1} = 0^3 + C \Rightarrow C = 1$$

$$\therefore \frac{1}{y-1} = x^3 + 1 \Rightarrow \boxed{y = 1 + \frac{1}{1+x^3}}$$

3. (15 points) Find the general solution for the following differential equation:

$$\frac{dy}{dx} + \frac{3x^2y}{1+x^3} = \frac{2x}{(1+x^2)(1+x^3)}$$

— (A)

THIS IS A 1st ORDER, LINEAR ODE IN STANDARD FORM \Rightarrow INTEGRATING FACTOR.

$$\therefore p(x) = \frac{3x^2}{1+x^3} \Rightarrow \int p(x) dx = \int \frac{3x^2}{1+x^3} dx$$

$$\boxed{u = 1+x^3 \\ du = 3x^2}$$

$$\begin{aligned} \therefore I(x) &= e^{\int p(x) dx} = e^{\ln(1+x^3)} \\ &= 1+x^3 \end{aligned}$$

$$= \int \frac{du}{u} = \ln u = \ln(1+x^3)$$

\therefore MULTIPLYING (A) BY $I(x) = (1+x^3)$

$$\frac{d}{dx} \left[(1+x^3)y \right] = (1+x^3) \frac{dy}{dx} + 3x^2 y = \frac{2x}{1+x^2}$$

$$\Rightarrow (1+x^3)y = \int \frac{2x}{1+x^2} + C$$

$$\begin{array}{l} v = 1+x^2 \\ dv = 2x \end{array}$$

$$= \int \frac{dv}{v} + C = \ln v + C$$

$$= \ln(1+x^2) + C$$

⇒

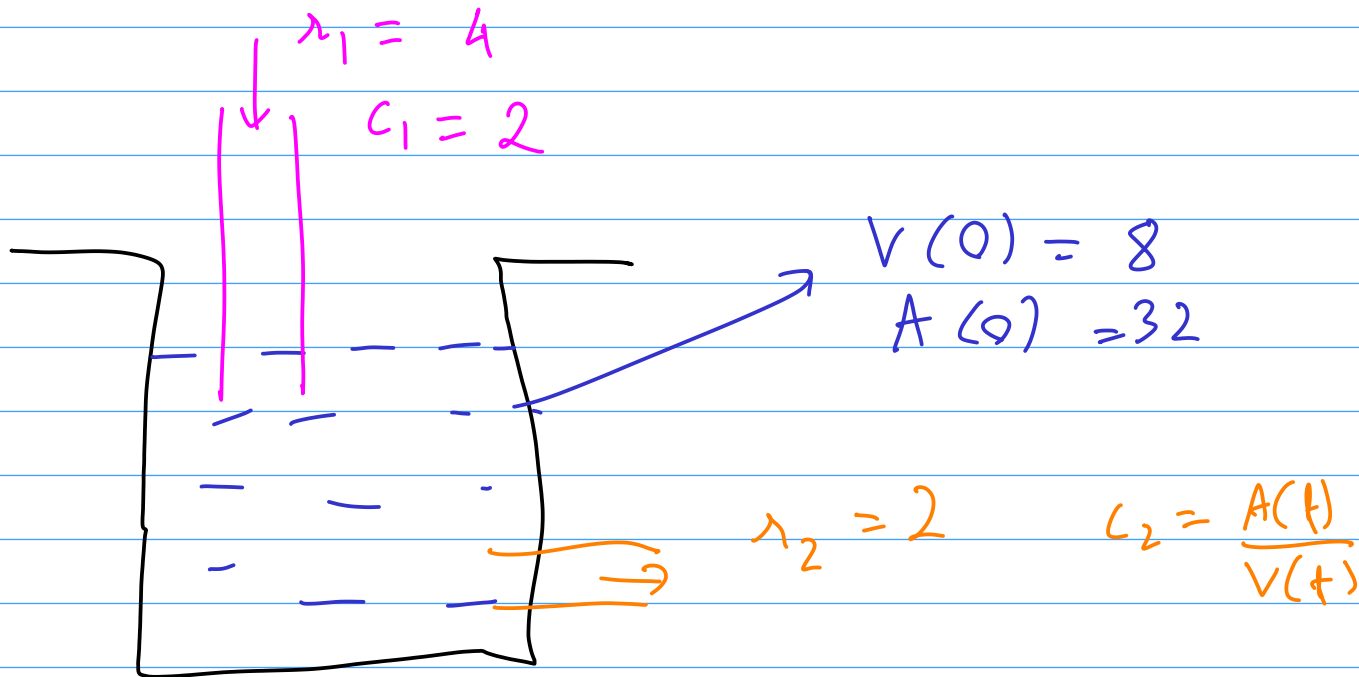
$$y = \frac{\ln(1+x^2) + C}{1+x^2}$$

4. (20 points) A tank contains 8L of water in which is dissolved 32g of chemical. A solution containing 2g/L of the chemical flows into the tank at a rate of 4L per minute and the well-stirred mixture flows out at a rate of 2L per minute.

- (a) Determine the amount of chemical in the tank after 20 minutes.
- (b) What is the concentration of chemical in the tank at that time?

$A(t)$ → AMOUNT
OF CHEMICAL

$V(t)$ → VOLUME



VOLUME
EQN

$$\begin{aligned} &: \frac{dV}{dt} = \text{VOLUME IN-FLOW} - \text{VOLUME OUT-FLOW} \\ &= 1 - 1 = 4 - 2 = 2 \end{aligned}$$

$$\Rightarrow \int_{V(0)}^{V(t)} dV = \int_0^t 2 dt$$

$$\Rightarrow V(t) - 8 = 2t$$

$$\Rightarrow \boxed{V(t) = 2t + 8}$$

$$\begin{array}{l} \text{AMOUNT :} \\ \text{EQN} \end{array} \quad \frac{dA}{dt} = \text{AMOUNT IN-FLOW} - \text{AMOUNT OUT-FLOW}$$

$$= r_1 c_1 - r_2 c_2$$

$$= (4)(2) - (2) \left(\frac{A}{V} \right)$$

$$= 8 - \frac{2A}{2t+8}$$

$$= 8 - \frac{A}{t+4}$$

$$\Rightarrow \frac{dA}{dt} + \frac{A}{t+4} = 8$$

$$\Rightarrow \frac{d}{dt} \left[(t+4) A(t) \right] = 8(t+4)$$

$$\Rightarrow \int_0^t \frac{d}{dt} \left[(t+4) A(t) \right] dt = \int_0^t 8(t+4) dt$$

$$\Rightarrow (t+4) A(t) - 4 A(0) = 4t^2 + 32t$$

$$\Rightarrow A(t) = \frac{4t^2 + 32t + 128}{t + 4}$$

$$\therefore (a) \quad A(20) = \frac{4 \times (20^2) + 32(20) + 128}{20 + 4}$$

$$= \boxed{\frac{296}{3} \text{ g}}$$

$$(b) \quad \frac{A(20)}{V(20)} = \frac{296/3}{2(20) + 4} = \frac{37}{18} \text{ g/L}$$

5. (25 points) Compute the rank of the following matrix by finding its reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{0} & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{P_{12}} \begin{bmatrix} \boxed{1} & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{A_3(-3)} \begin{bmatrix} \boxed{1} & 2 & 3 & 2 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & -5 & -8 & -3 \end{bmatrix}$$

$$\begin{array}{l}
 A_{21}(-2) \\
 \longrightarrow \\
 A_{23}(5)
 \end{array}
 \begin{bmatrix}
 \boxed{1} & 0 & -1 & 0 \\
 0 & \boxed{1} & 2 & 1 \\
 0 & 0 & \boxed{2} & 2
 \end{bmatrix}
 \xrightarrow{M_3(\frac{1}{2})}
 \begin{bmatrix}
 \boxed{1} & 0 & -1 & 0 \\
 0 & \boxed{1} & 2 & 1 \\
 0 & 0 & \boxed{1} & 1
 \end{bmatrix}$$

$$\begin{array}{l}
 A_{31}(1) \\
 A_{32}(-2)
 \end{array}
 \begin{bmatrix}
 1 & 0 & 0 & 1 \\
 0 & 1 & 0 & -1 \\
 0 & 0 & 1 & 1
 \end{bmatrix}$$

CLEARLY, # OF NON-ZERO ROWS = 3

$\therefore \text{RANK} = 3$

6. (20 points) Use Gaussian elimination or Gauss-Jordan elimination to solve the following system of equations:

$$x_1 + 2x_2 + x_3 = 9,$$

$$3x_1 + 5x_2 + x_3 = 18,$$

$$2x_1 + 6x_2 + 7x_3 = 40.$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 1 \\ 2 & 6 & 7 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 9 \\ 18 \\ 40 \end{bmatrix}$$

$$\therefore A^{\#} = \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 9 \\ 3 & 5 & 1 & 18 \\ 2 & 6 & 7 & 40 \end{array} \right]$$

$A_{12}(-3)$

$\downarrow A_{13}(-2)$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 9 \\ 0 & \boxed{-1} & -2 & -9 \\ 0 & 2 & 5 & 22 \end{array} \right] \xrightarrow{M_2(-1)} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 9 \\ 0 & \boxed{1} & 2 & 9 \\ 0 & 2 & 5 & 22 \end{array} \right]$$



$\downarrow A_{21}(-2), A_{23}(-2)$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -9 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

$$\therefore x_1 - 3x_3 = -9$$

$$x_2 + 2x_3 = 9$$

$$x_3 = 4$$

$$\Rightarrow x_3 = 4, \quad x_2 = 9 - 2x_3 = 1, \quad x_1 = 3x_3 - 9 = 3$$

$$\therefore (x_1, x_2, x_3) = (3, 1, 4)$$

