MATH 165 (SUMMER '22, SESH B2)

MIDTERM 1

SOLUTIONS

(EVENING SLOT)

2. (20 points)

Consider the differential equation

$$y' = -3x^2(y+1)^2.$$

(a) What are the isoclines for this differential equation?

LET $k \in \mathbb{R}$ BE FIXED AND ARBITRARY.

$$y' = f(x,y) = -3x^2(y+1)^2$$

ISOCLINES: $f(x,y) = k \rightarrow \boxed{-3x^2(y+1)^2 = k}$

(b) Are there any equilibrium solutions? Identify them.

AN EQ. SOLN IS $Y = c$ s.t. $f(x,y) \equiv 0$.

$$f(x,y) = -3x^2 (y+1)^2 = 0$$

$$\Rightarrow \quad x = 0 \quad \text{OR} \quad \underbrace{y+1 = 0}_{\Downarrow} \quad \boxed{y = -1}$$

NOT AN EQ. SOLN. (∵ NOT OF THE FORM $Y = c$)

YES!

(c) Solve the initial value problem given by this differential equation together with the condition $y(0) = 0$.

$$y' = -3x^2(y+1)^2$$

SEPARABLE ! $\longrightarrow$ $$\frac{-dy}{(y+1)^2} = 3x^2\,dx$$

$$\Rightarrow \quad -\int \frac{dy}{(y+1)^2} = \int 3x^2\,dx + C$$

$$\Rightarrow \quad \frac{1}{y+1} = x^3 + C$$

$$y(0) = 0 \Rightarrow \frac{1}{0+1} = 0^3 + C \Rightarrow C = 1$$

$$\therefore \frac{1}{y+1} = x^3 + 1 \Rightarrow \boxed{y = \frac{1}{1+x^3} - 1}$$

3. (15 points) Find the general solution for the following differential equation:

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{3x^2}{(1+x^2)(1+x^3)}.$$

$-$ B

THIS IS A 1st ORDER, LINEAR ODE IN STANDARD FORM $\Rightarrow$ INTEGRATING FACTOR.

$\therefore p(x) = \dfrac{2x}{1+x^2} \Rightarrow \displaystyle\int p(x)\,dx = \int \frac{2x}{1+x^2}\,dx$

$\boxed{\begin{matrix} u = 1+x^2 \\ du = 2x \end{matrix}}$

$= \displaystyle\int \frac{du}{u} = \ln u = \ln(1+x^2)$

$\therefore I(x) = e^{\int p(x)dx} = e^{\ln(1+x^2)}$

$= 1+x^2$

$\therefore$ MULIPYING (A) BY $I(x) = (1+x^3)$

$$\frac{d}{dx}\left[(1+x^2)y\right] = (1+x^2)\frac{dy}{dx} + 2xy = \frac{3x^2}{1+x^3}$$

$$\Rightarrow (1+x^2)y = \int \frac{3x^2\,dx}{1+x^3} + C$$

$$\boxed{\begin{aligned} v &= 1+x^3 \\ dv &= 3x^2 \end{aligned}}$$

$$= \int \frac{dv}{v} + C = \ln v + C$$

$$= \ln(1+x^3) + C$$

$$\Rightarrow \boxed{y = \frac{\ln(1+x^3) + C}{1+x^2}}$$

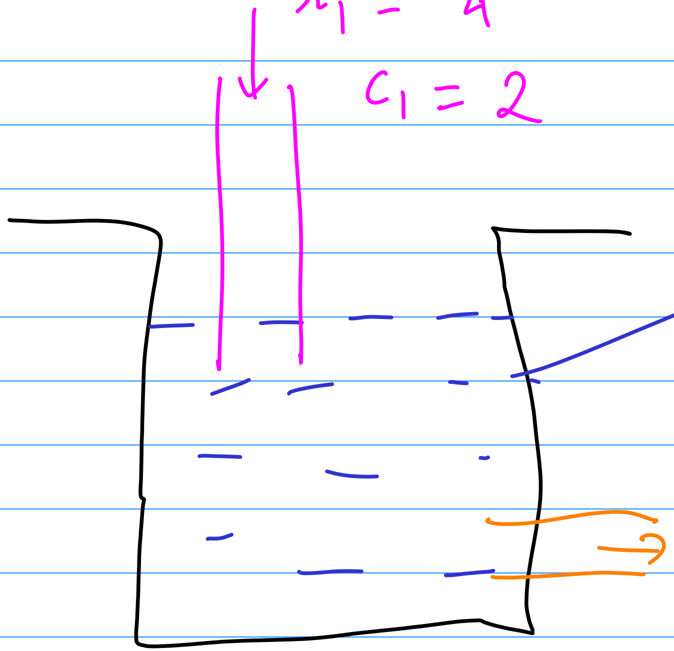
4. (20 points) A tank contains $8L$ of water in which is dissolved $32g$ of chemical. A solution containing $2g/L$ of the chemical flows into the tank at a rate of $4L$ per minute and the well-stirred mixture flows out at a rate of $2L$ per minute.

(a) Determine the amount of chemical in the tank after 20 minutes.

(b) What is the concentration of chemical in the tank at that time?

$r_1 = 4$

$c_1 = 2$



$V(0) = 8$

$A(0) = 32$

$A(t) \to$ AMOUNT OF CHEMICAL

$V(t) \to$ VOLUME

$r_2 = 2$

$c_2 = \frac{A(t)}{V(t)}$

VOLUME EQN :

$$\frac{dV}{dt} = \text{VOLUME IN-FLOW} - \text{VOLUME OUT-FLOW}$$

$$= r_1 - r_2 = 4 - 2 = 2$$

$$\Rightarrow \int_{V(0)}^{V(t)} dV = \int_0^t 2\,dt$$

$$\Rightarrow V(t) - 8 = 2t$$

$$\Rightarrow \boxed{V(t) = 2t + 8}$$

AMOUNT EQN : $\frac{dA}{dt}$ = AMOUNT IN-FLOW − AMOUNT OUT-FLOW

$$= r_1 c_1 - r_2 c_2$$

$$= (4)(2) - (2)\left(\frac{A}{V}\right)$$

$$= 8 - \frac{2A}{2t+8}$$

$$= 8 - \frac{A}{t+4}$$

$$\Rightarrow \frac{dA}{dt} + \frac{A}{t+4} = 8$$

$$\Rightarrow \frac{d}{dt}\left[(t+4)\,A(t)\right] = 8(t+4)$$

$$\Rightarrow \int_0^t \frac{d}{dt}\left[(t+4)\,A(t)\right] dt = \int_0^t 8\,(t+4)\,dt$$

$$\Rightarrow (t+4)\,A(t) - 4\,A(0) = 4t^2 + 32t$$

$$\Rightarrow \quad A(t) = \frac{4t^2 + 32t + 128}{t+4}$$

$$\therefore \text{(a)} \quad A(20) = \frac{4 \times (20^2) + 32(20) + 128}{20+4}$$

$$= \boxed{\frac{296}{3} \text{ g}}$$

$$\text{(b)} \quad \frac{A(20)}{V(20)} = \frac{296/3}{2(20)+4} = \frac{37}{18} \text{ g/L}$$

5. (25 points) Compute the rank of the following matrix by finding its reduced row echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{bmatrix} \xrightarrow[A_{13}(-2)]{A_{12}(-3)} \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$$\Big\downarrow A_{21}(-1),\ A_{23}(-1)$$

$$\begin{bmatrix} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \xleftarrow{M_3(-1/2)} \begin{bmatrix} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 0 & \boxed{-2} \end{bmatrix}$$

$$\Big\downarrow A_{32}(1),\ A_{32}(-1)$$

$$\begin{bmatrix} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

CLEARLY, # OF NON-ZERO ROWS = 3

$\therefore$ RANK = 3

6. (20 points) Use Gaussian elimination or Gauss-Jordan elimination to solve the following system of equations:

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 17, \\
3x_1 + 5x_2 + x_3 &= 42, \\
2x_1 + 6x_2 + 7x_3 &= 53.
\end{aligned}
$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 1 \\ 2 & 6 & 7 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 17 \\ 42 \\ 53 \end{bmatrix}$$

$$\therefore\ A^{\#} = \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 17 \\ 3 & 5 & 1 & 42 \\ 2 & 6 & 7 & 53 \end{array}\right]$$

$$\Big\downarrow A_{12}(-3),\ A_{13}(-2)$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 17 \\ 0 & \boxed{-1} & -2 & -9 \\ 0 & 2 & 5 & 19 \end{array}\right] \xrightarrow{M_2(-1)} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 17 \\ 0 & \boxed{1} & 2 & 9 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

$$\Big\downarrow$$

$$\downarrow A_{21}(-2),\ A_{23}(-2)$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -3 & -1 \\ 0 & \boxed{1} & 2 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\therefore \quad \begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 + 2x_3 &= 9 \\ x_3 &= 1 \end{aligned}$$

$\Rightarrow x_3 = 1, \quad x_2 = 9 - 2x_3 = 7, \quad x_1 = 3x_3 - 1 = 2$

$$\therefore \quad (x_1, x_2, x_3) = (2, 7, 1)$$

$$(e = 2.71\ldots)$$