# MATH 165 (SUMMER '22, SESH B2)

## MIDTERM 2

## SOLUTIONS

(MORNING SLOT)

**2. (25 points)**

Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \\ 1 & \frac{1}{2022} & 2022 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 2 & 2 & 1 \\ -2 & 0 & 4 & 1 \\ 0 & -2 & 3 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} -3 & 5 & 6 & -14 \\ 0 & 2 & 13 & -156 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

(a) Find the determinants of $A, B, C$. Which of $A, B, C$ are invertible?

THIS IS THE SAME AS SAMPLE MIDTERM, Q2 (a).

$\det A = 0$

$\det B = 3$

$\det C = 10$

(b) Find the determinant of $AB^3$.

$$\det(AB^3) = (\det A)\left[\det(B^3)\right]$$

$$= 0 \times \left[\det(B^3)\right] = 0$$

(c) Find the determinant of $C^T B$.

$$\det(C^T B) = (\det C^T)(\det B) = (\det C)(\det B)$$

$$= (10)(3) = 30$$

**3. (20 points)** For the following choices of $S$ and $V$, determine whether $S$ is a subspace of $V$.

(a)

$$V = M_2(\mathbb{R}), S = \{A \in M_2(\mathbb{R}) : \det A = 0.\}$$

LET $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\det A_1 = \det A_2 = 0 \Rightarrow A_1, A_2 \in S$

BUT, $\det(A_1 + A_2) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$

$\Rightarrow A_1 + A_2 \in S \Rightarrow S$ IS <u>NOT</u> A SUBSPACE

($\because$ NOT CLOSED UNDER $+$)

(b)

$$V = \cancel{P_2(\mathbb{R})}\ P_5(\mathbb{R}),\ S = \{p(x) \in P_5(\mathbb{R}) : p'(0) = 1\}.$$

LET $q(x) = 0$ [ZERO VECTOR]

$\Rightarrow q'(0) = 0 \neq 1 \Rightarrow q \notin S$

$\Rightarrow$ S IS NOT A SUBSPACE

[$\because$ ZERO VECTOR CHECK]

(c)

$$V = \mathbb{R}^5, S = \{(x_1, x_2, x_3, x_4, x_5) : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}.$$

THIS IS A SUBSPACE.

Pf ① CLOSURE UNDER +.

LET $\vec{x} = (x_1, \ldots, x_5)$, $\vec{y} = (y_1, \ldots, y_5)$

s.t. $\vec{x}, \vec{y} \in S$. $\Rightarrow$

$x_1 + x_2 + x_3 + x_4 + x_5 = y_1 + y_2 + y_3 + y_4 + y_5 = 0$

$\vec{z} = \vec{x} + \vec{y}$

$$\therefore \quad (z_1, z_2, z_3, z_4, z_5) = (x_1, x_2, x_3, x_4, x_5)$$

$$+ (y_1, y_2, y_3, y_4, y_5)$$

$$= (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5)$$

$$\Rightarrow z_1+z_2+z_3+z_4+z_5 = (x_1+y_1)+(x_2+y_2)+(x_3+y_3)+(x_4+y_4)$$

$$+ (x_5+y_5)$$

$$= (x_1+x_2+x_3+x_4+x_5)$$

$$+ (y_1+y_2+y_3+y_4+y_5)$$

$$= 0 + 0 = 0 \qquad \Rightarrow \vec{x}+\vec{y} \in S$$

② CLOSURE UNDER SCALAR MULTIPLICATION.

$c \in \mathbb{R}, \quad \vec{x} = (x_1, x_2, x_3, x_4, x_5) \in S \quad [\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 = 0]$

$\vec{y} = c\vec{x} \Rightarrow (y_1, y_2, y_3, y_4, y_5) = c(x_1, x_2, x_3, x_4, x_5)$

$= (cx_1, cx_2, cx_3, cx_4, cx_5)$

$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 = cx_1 + cx_2 + cx_3 + cx_4 + cx_5$

$= c[x_1 + x_2 + x_3 + x_4 + x_5] = c[0] = 0 \Rightarrow c\vec{x} \in S$

**4.** **(20 points)** For the following choices of finite sets $S$, determine whether $S$ spans $V$, whether $S$ is linearly independent in $V$, and whether $S$ is a basis for $V$.

(a) $V = P_3(\mathbb{R})$ with $S = \{-3x^2 + x + 4, 9x^3 + 7x^2 - 5x + 7, 2x^3 + 2x + 2\}$.

$\dim V = \dim P_3(\mathbb{R}) = 3 + 1 = 4$

$|S| = 3$

$|S| < \dim V \Rightarrow S$ IS NOT SPANNING.

NOW LET $c_1(-3x^2 + x + 4) + c_2(9x^3 + 7x^2 - 5x + 7)$

$+ c_3(2x^3 + 2x + 2) = 0$

$$\Rightarrow \quad x^3(2c_3 + 9c_2) + x^2(7c_2 - 3c_1) + x(c_1 - 5c_2 + 2c_3) + (4c_1 + 7c_2 + 2c_3) = 0$$

$\therefore$

$$\left.\begin{array}{rcrcrl} & & 9c_2 & + & 2c_3 & = 0 \\ -3c_1 & + & 7c_2 & & & = 0 \\ c_1 & - & 5c_2 & + & 2c_3 & = 0 \\ 4c & - & 7c_2 & + & 2c_3 & = 0 \end{array}\right\} \qquad A\vec{c} = \vec{0}$$

$$A = \begin{bmatrix} 0 & 9 & 2 \\ -3 & 7 & 0 \\ 1 & -5 & 2 \\ 4 & -7 & -2 \end{bmatrix}$$

BY GAUSS-JORDAN ELIMINATION,

$$\begin{bmatrix} 0 & 9 & 2 \\ -3 & 7 & 0 \\ 1 & -5 & 2 \\ 4 & -7 & -2 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad c_1 = c_2 = c_3 = 0$

$\therefore$ LINEARLY INDEPENDENT.

(b) $V = \mathbb{R}^4$ with $S = \{(0,0,1,1), (0,1,0,1), (0,1,-1,0), (1,0,0,1), (2,-1,-1,0)\}$.

$$\left.\begin{array}{l} \dim V = \dim \mathbb{R}^4 = 4 \\ |S| = 5 \end{array}\right\} \Rightarrow \quad S \text{ IS NOT L.I.}$$

NOTE THAT $(2,-1,-1,0) = 2(1,0,0,1) - (0,0,1,1) - (0,1,0,1)$

& $(0,1,-1,0) = (0,1,0,1) - (0,0,1,1)$

$$\therefore \text{SPAN}(S) = \text{SPAN}\underbrace{\{(0,0,1,1), (0,1,0,1), (1,0,0,1)\}}_{S'}$$

$\because \ |S'| = 3 < 4 = \dim V$

$\Rightarrow$ $S'$ IS NOT SPANNING

$\Rightarrow$ $S$ IS NOT SPANNING.

(c) $V = M_{2\times 3}(\mathbb{R})$ with

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

NOTE, $\dim V = \dim M_{2\times 3}(\mathbb{R}) = 2\times 3 = 6$

$|S| = 6$

$\therefore |S| = \dim V \Rightarrow$ (SPANNING $\Leftrightarrow$ L.I.)

$\therefore$ SUFFICES TO SHOW THAT $S$ IS L.I.

NOW, LET

$$c_1\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}+c_2\begin{bmatrix}1&0&0\\1&0&0\end{bmatrix}+c_3\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}$$

$$+c_4\begin{bmatrix}0&1&0\\0&1&0\end{bmatrix}+c_5\begin{bmatrix}0&0&0\\0&1&1\end{bmatrix}+c_6\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}c_1+c_2+c_3 & c_3+c_4+c_6 & c_6\\ c_2 & c_4+c_5 & c_5\end{bmatrix}=0$$

$\therefore$ $c_1+c_2+c_3=0$, $c_3+c_4+c_6=0$, $c_6=0$, $c_2=0$, $c_4+c_5=0$, $c_5=0$

$\Rightarrow$ $c_1=c_2=c_3=c_4=c_5=0$ (BACK SUBST.)

**5. (35 points)** Consider the matrix

$$\begin{bmatrix} 2 & 9 & -1 & 3 \\ 1 & 4 & -1 & 1 \\ 1 & 5 & 0 & 2 \\ 0 & 3 & 3 & 3 \end{bmatrix}.$$

CHECK THAT

$$\begin{bmatrix} 2 & 9 & -1 & 3 \\ 1 & 4 & -1 & 1 \\ 1 & 5 & 0 & 2 \\ 0 & 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} 1 & 0 & -5 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(ROW)

(COLUMN)

(a) Find a basis for the row-space, and hence compute the dimension of the row-space.

BASIS = ROWS IN ROW-ECHELON FORM

$= \{ (1, 0, -5, -3), (0, 1, 1, 1) \}$ , DIM = 2

(b) Find a basis for the column-space, and hence compute the dimension of the column-space.

BASIS = COLUMNS IN THE ORIGINAL MATRIX CORRESPONDING TO LEADING 1s IN ROW-ECHELON FORM

$= \{ (2, 1, 1, 0), (9, 4, 5, 3) \}$ , DIM = 2

(c) Find a basis for the null-space, and hence compute the dimension of the null-space.

BY REF,

$$x_1 - 5x_3 - 3x_4 = 0$$

$$x_2 + x_3 + x_4 = 0$$

LET $x_3 = s$, $x_4 = t$

$$\Rightarrow x_1 = 5s + 3t$$

$$x_2 = -s - t$$

$$\therefore \vec{x} = (x_1, x_2, x_3, x_4) = (5s + 3t, -s + t, s, t)$$

$$= (5s, -s, s, 0) + (3t, t, 0, t)$$

$$= s(5, -1, 1, 0) + t(3, 0, 0, 1)$$

$\therefore$ BASIS $= \{(5, -1, 1, 0), (3, 0, 0, 1)\}$

$\therefore$ DIM $= 4$

(d) Do your answers in the previous parts match your expectation? Explain.

→ DIM (ROW-SPACE) = 2 = DIM (COL-SPACE) [= RANK]

→ $\text{RANK} + \underbrace{\text{NULLITY}}_{\text{DIM(NULLSPACE)}} = 2 + 2 = 4$

= # OF COLUMNS

AS EXPECTED !