MATH 165 (SUMMER '22, SESH B2)

MIDTERM 2

SOLUTIONS

(EVENING SLOT)

2. (25 points)

Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \\ 1 & \frac{1}{2022} & 2022 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 2 & 2 & 1 \\ -2 & 0 & 4 & 1 \\ 0 & -2 & 3 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} -3 & 5 & 6 & -14 \\ 0 & 2 & 13 & -156 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

(a) Find the determinants of A, B, C. Which of A, B, C are invertible?

THIS IS THE SAME AS SAMPLE MIDTERM, Q2 (a).

$\det A = 0$

$\det B = 3$

$\det C = 10$

(b) Find the determinant of A^2BC .

$$\det(A^2BC) = \det(A \cdot ABC) = (\det A)(\det ABC)$$

$$= 0 \cdot (\det ABC) = 0$$

(c) Find the determinant of B^TC.

$$\det B^TC = (\det B^T)(\det C) = (\det B)(\det C)$$

$$= (3)(10)$$

$$= 30$$

3. (20 points) For the following choices of S and V, determine whether S is a subspace of V.

(a)

$$V = M_2(\mathbb{R}), S = \{A \in M_2(\mathbb{R}) : \det A = -1.\}$$

LET $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \det A = 0 \neq -1$

$\Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin S$

$\Rightarrow$ S IS NOT A SUBSPACE

(ZERO VECTOR CHECK)

(b)

$$V = \cancel{P_2}P_5(\mathbb{R}), S = \{p(x) \in P_5(\mathbb{R}) : p(0)p(1) = 0\}.$$

CONSIDER $p_1(x) = x$

$p_2(x) = x - 1$

$\therefore\ p_1(0)\,p_1(1) = (0)(1) = 0$

& $p_2(0)\,p_2(1) = (0-1)(1-1) = 0$

$\therefore\ p_1, p_2 \in S$

BUT, IF $p = p_1 + p_2$, $p(x) = 2x - 1$

$\Rightarrow p(0) = -1,\ p(1) = 1 \Rightarrow p(0)\,p(1) = -1 \neq 0 \Rightarrow$ NOT CLOSED UNDER $+$!

(c)

$$V = \mathbb{R}^5, S = \{(x_1, x_2, x_3, x_4, x_5) : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}.$$

THIS IS A SUBSPACE.

Pf ① CLOSURE UNDER $+$.

LET $\vec{x} = (x_1, \ldots, x_5)$, $\vec{y} = (y_1, \ldots, y_5)$

s.t. $\vec{x}, \vec{y} \in S$. $\Rightarrow$

$x_1 + x_2 + x_3 + x_4 + x_5 = y_1 + y_2 + y_3 + y_4 + y_5 = 0$

$\vec{z} = \vec{x} + \vec{y}$

$$\therefore \quad (z_1, z_2, z_3, z_4, z_5) = (x_1, x_2, x_3, x_4, x_5)$$
$$+ (y_1, y_2, y_3, y_4, y_5)$$
$$= (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5)$$
$$\Rightarrow z_1+z_2+z_3+z_4+z_5 = (x_1+y_1)+(x_2+y_2)+(x_3+y_3)+(x_4+y_4)$$
$$+ (x_5+y_5)$$
$$= (x_1+x_2+x_3+x_4+x_5)$$
$$+ (y_1+y_2+y_3+y_4+y_5)$$
$$= 0 + 0 = 0 \qquad \Rightarrow \vec{x}+\vec{y} \in S$$

② CLOSURE UNDER SCALAR MULTIPLICATION.

$c \in \mathbb{R}, \quad \vec{x} = (x_1, x_2, x_3, x_4, x_5) \in S \quad [\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 = 0]$

$\vec{y} = c\vec{x} \Rightarrow (y_1, y_2, y_3, y_4, y_5) = c(x_1, x_2, x_3, x_4, x_5)$

$= (cx_1, cx_2, cx_3, cx_4, cx_5)$

$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 = cx_1 + cx_2 + cx_3 + cx_4 + cx_5$

$= c[x_1 + x_2 + x_3 + x_4 + x_5] = c[0] = 0 \Rightarrow c\vec{x} \in S$

4. (20 points) For the following choices of finite sets S, determine whether S spans V, whether S is linearly independent in V, and whether S is a basis for V.

(a) $V = P_3(\mathbb{R})$ with $S = \{x+1, x^2+1, x^2-x, x^3+1, 2x^3-x^2-x\}$.

$$\left.\begin{array}{l} \dim V = \dim P_3(\mathbb{R}) = 3+1 = 4 \\ |S| = 5 \end{array}\right\} \Rightarrow S \text{ IS NOT L.I.}$$

OTOH,

$$x^2 - x = (x^2+1) - (x+1)$$

$$2x^3 - x^2 - x = 2(x^3+1) - (x^2+1) - (x+1)$$

$$\text{SPAN}(S) = \text{SPAN}\{\underbrace{x+1, x^2+1, x^3+1}_{S'}\}$$

$\because \; |S'| = 3 < 4 = \dim V$

$\Rightarrow$ S' IS NOT SPANNING

$\Rightarrow$ S IS NOT SPANNING.

(b) $V = \mathbb{R}^4$ with $S = \{(0, -3, 1, 4), (9, 7, -5, 7), (2, 0, 2, 2)\}$.

$$\left.\begin{array}{l} \dim V = \dim \mathbb{R}_4 = 4 \\ |S| = 3 \end{array}\right\} \Rightarrow$$ S IS NOT SPANNING.

NOW LET $c_1(0, -3, 1, 4) + c_2(9, 7, -5, 7) + c_3(2, 0, 2, 2) = 0$

$\Rightarrow (2c_3 + 9c_2, 7c_2 - 3c_1, c_1 - 5c_2 + 2c_3, 4c_1 + 7c_2 + 2c_3)$

$\therefore$

$$\left.\begin{aligned}
9c_2 + 2c_3 &= 0 \\
-3c_1 + 7c_2 &= 0 \\
c_1 - 5c_2 + 2c_3 &= 0 \\
4c - 7c_2 + 2c_3 &= 0
\end{aligned}\right\}$$

$$A\vec{c} = \vec{0}$$

$$A = \begin{bmatrix} 0 & 9 & 2 \\ -3 & 7 & 0 \\ 1 & -5 & 2 \\ 4 & -7 & -2 \end{bmatrix}$$

BY GAUSS-JORDAN ELIMINATION,

$$\begin{bmatrix} 0 & 9 & 2 \\ -3 & 7 & 0 \\ 1 & -5 & 2 \\ 4 & -7 & -2 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad c_1 = c_2 = c_3 = 0$

$\therefore$ LINEARLY INDEPENDENT.

(c) $V = M_{3\times 2}(\mathbb{R})$ with

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

NOTE, $\dim V = \dim M_{3\times 2}(\mathbb{R}) = 3\times 2 = 6$

$|S| = 6$

$\therefore |S| = \dim V \Rightarrow$ (SPANNING $\Leftrightarrow$ L.I.)

$\therefore$ SUFFICES TO SHOW THAT S IS L.I.

NOW, LET

$$c_1\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}+c_2\begin{bmatrix}1&1\\0&0\\0&0\end{bmatrix}+c_3\begin{bmatrix}1&0\\1&0\\0&0\end{bmatrix}+c_4\begin{bmatrix}0&0\\1&1\\0&0\end{bmatrix}+c_5\begin{bmatrix}0&0\\0&1\\0&1\end{bmatrix}+c_6\begin{bmatrix}0&0\\1&0\\1&0\end{bmatrix}=0$$

$$\Rightarrow \begin{bmatrix}c_1+c_2+c_3 & c_2\\ c_3+c_4+c_6 & c_4+c_5\\ c_6 & c_5\end{bmatrix}=0$$

$\therefore$ $c_1+c_2+c_3=0$, $c_3+c_4+c_6=0$, $c_6=0$, $c_2=0$,
$c_4+c_5=0$, $c_5=0$

$\Rightarrow c_1=c_2=c_3=c_4=c_5=0$ (BACK SUBST.)

5. (35 points) Consider the matrix

$$\begin{bmatrix} 2 & 9 & -1 & 3 \\ 1 & 4 & -1 & 3 \\ 1 & 5 & 0 & 2 \\ 0 & 3 & 3 & 1 \end{bmatrix}.$$

CHECK THAT

$$\begin{bmatrix} 2 & 9 & -1 & 3 \\ 1 & 4 & -1 & 3 \\ 1 & 5 & 0 & 2 \\ 0 & 3 & 3 & 1 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(ROW)

(COLUMN)

(a) Find a basis for the row-space, and hence compute the dimension of the row-space.

BASIS = ROWS IN ROW-ECHELON FORM

$= \{(1,0,-5,0), (0,1,1,0), (0,0,0,1)\},$ DIM=3

(b) Find a basis for the column-space, and hence compute the dimension of the column-space.

BASIS = COLUMNS IN THE ORIGINAL MATRIX CORRESPONDING TO LEADING 1s IN ROW-ECHELON FORM

$= \{(2,1,1,0), (9,4,5,3), (3,3,2,1)\}$

DIM = 3

(c) Find a basis for the null-space, and hence compute the dimension of the null-space.

BY REF,

$$x_1 - 5x_3 = 0$$

$$x_2 + x_3 = 0$$

$$x_4 = 0$$

LET $x_3 = t \Rightarrow x_1 = 5t$, $x_2 = -t$

$$\therefore \vec{x} = (x_1, x_2, x_3, x_4) = (5t, -t, t, 0)$$

$$= t(5, -1, 1, 0)$$

$\therefore$ BASIS $= \{(5, -1, 10)\}$

$\therefore$ DIM $= 1$

(d) Do your answers in the previous parts match your expectation? Explain.

→ DIM (ROW-SPACE) = 3 = DIM (COL-SPACE) [= RANK]

→ RANK + $\underbrace{\text{NULLITY}}_{\text{DIM(NULLSPACE)}}$ = 3 + 1 = 4

= # OF COLUMNS

AS EXPECTED !