Homework 10 (Extra Credit)

MATH 201 (Summer 2023, Session A2)

Sunday 11$^{\text{th}}$ June, 2023

Instructions

- This homework is due on Wednesday, June 21st at 11 PM Eastern Time.
- This homework is for extra credit. Please see the grading policy of the course for more details on how this extra credit will be incorporated into your score.
- Justify your answers.
- Late submissions are not permitted for this homework. If you are unable to submit this homework on time due to extenuating circumstances, please contact the instructor and ask to set up an incomplete contract.
- Please read the honesty policy of the course (available on the course webpage) and make sure you understand the collaboration policy.

Problem 0. [0 points] Copy paste the following text in the beginning of your submission:

> This submission conforms to the honesty policy of the course. In particular, I have not made use of any unauthorized online resources and any collaboration did not violate the expectations outlined in the policy.

After that, list all students you collaborated with, clearly indicating which problems you worked with them on. If you did not collaborate with anyone, clearly state this instead.

Problem 1. [20 points] I have picked two distinct integers[1], x and y, written them on two sheets of paper, and then put them into envelopes without showing them to you. Your goal is to pick the envelope with the higher integer. Due to symmetry, it is clear that the optimal strategy here is to pick one of the envelopes uniformly at random, and hence pick the higher integer with probability 1/2.
After you make a choice, I let you open the envelope and see the number. You are then given a choice: you can either keep the envelope you chose, or you can swap to the other envelope. Show that there is a randomized strategy you may employ such that for any choice of x, y above, there exists an $\epsilon = \epsilon(x, y) > 0$ so that the probability of picking the higher envelope

[1] possibly negative

under this strategy is $1/2+\epsilon$. [**Hint:** Without loss of generality, suppose the integer revealed is x. Pick a real number r according to the normal distribution[2]. Now, check whether $r < x$ or $r \geq x$ and decide whether to change envelopes accordingly.]

Problem 2. [10 points]

(a) Let $A \sim \mathrm{Exp}(\lambda)$ and $B \sim \mathrm{Exp}(\lambda)$ be independent. In particular, this means that $A \geq 0$ with probability 1, and hence $\sqrt{A}$ is well-defined. What is the probability that

$$x^2 + \sqrt{A}x + B,$$

has two real roots?

(c) Let $(C, D) \sim \mathrm{Unif}[0,1]^2$. What is the probability that

$$x^2 + Cx + D,$$

has no real roots?

Problem 3. [20 points] Let $Z_1, \dots, Z_k$ be independent and standard normal random variables and define

$$Y = Z_1^2 + \cdots + Z_k^2.$$

The distribution of Y is called the χ_k^2-distribution; we considered $k = 1$ in the Sample Midterm.

(a) Find the moment generating function of Z_1^2. [**Hint:** use the definition $M_{Z_1^2}(t) = E[e^{tZ_1^2}]$ and be careful integrating, as $M_{Z_1^2}(t)$ is only finite for $t < 1/2$].

(b) Compute the mean and variance of Y.

(c) Find the moment generating function $M_Y(t)$ of Y.

(d) Compute $E[Y^3]$.

[2] Actually, any continuous distribution which is supported on all of $\mathbb{R}$ will work, i.e., provided that the p.d.f. $p(x) \neq 0$ for any $x \in \mathbb{R}$.