MATH 201 HW 10

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Let t be the random element which is takes values $\{x, y\}$ with probability 1/2 each. One element is definitely larger than the other. Without loss of generality assume that $x < y$. Sampling r from $\mathbb{Z}$, we can consider $P(r < x)$ and $P(r \geq x) = 1 - P(r < x)$. If $t = x$, and $r \geq x$, then our plan is to switch to y. The same strategy will be implemented if $t = y$. Therefore if x is chosen, there is a probability of $1 - P(r < x)$ to switch. Let A be the event where a correct guess is made.

$$P(A) = P(A|t = x)P(t = x) + P(A|t = y)P(t = y)$$

$$= P(r \geq x)\frac{1}{2} + P(r < y)\frac{1}{2} = (1 - P(r < x))\frac{1}{2} + P(r < y)\frac{1}{2}$$

This can be rewritten as

$$\frac{1}{2} + \frac{1}{2}(P(r < y) - P(r < x)) = \frac{1}{2} + \frac{1}{2}P(x < r < y)$$

Since P is a continuous probability distribution, this probability will be non-zero, so we can let $\varepsilon(x, y) = \frac{1}{2}P(x < r < y)$. This gives $P(A) = \frac{1}{2} + \varepsilon$.

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a) In order for a polynomial $ax^2 + bx + c$ to have two real roots, the discriminant $b^2 - 4ac > 0$. In our case, we require that $A - 4B > 0$. We then compute

$$P(A > 4B) = \int_0^\infty \int_{4y}^\infty \lambda^2 e^{-\lambda(x+y)} dxdy = \int_0^\infty -\lambda e^{-\lambda(x+y)}\Big|_{4y}^\infty dy = \int_0^\infty \lambda e^{-5y\lambda} dy$$

This then simplifies to

$$P(A > 4B) = -\frac{e^{-5y\lambda}}{5}\Big|_0^\infty = \frac{1}{5}$$

b) In order for a polynomial $ax^2 + bx + c$ to have no real roots, $b^2 - 4ac < 0$. In our case we require that $C^2 - 4D < 0$. We then find

$$P(C^2 - 4D < 0) = P(C^2/4 < D) = \int_0^1 \int_{y^2/4}^1 dxdy = \int_0^1 1 - y^2/4 \; dy$$

This becomes

$$P(C^2 - 4D < 0) = y - \frac{y^3}{12}\Big|_0^1 = 1 - \frac{1}{12} = \frac{11}{12}$$

3

a) We have that

$$M_{Z_1^2}(t) = E[e^{tZ_1^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{tz^2} e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{z^2(t-1/2)} dz$$

This becomes (assuming that $t < 1/2$).

$$\frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{1/2 - t}} = \frac{1}{\sqrt{1-2t}}$$

b) We can find the mean and variance of Z_1^2. Since Y is the sum of independent random variables, we just have to sum over the mean/variance of each Z_i.

$$E[Z_1^2] = \frac{d}{dt} M_{Z_1^2}(t)\bigg|_{t=0}$$

Then,

$$\frac{d}{dt} M_{Z_1^2}(t) = \frac{1}{(1-2t)^{3/2}}$$

This is equal to 1 when $t = 0$. Therefore $E[Z_1^2] = 1$. For the variance, we have to find

$$E[Z_1^4] = \frac{d^2}{dt^2} M_{Z_1^2}(t)\bigg|_{t=0} = \frac{3}{(1-2t)^{5/2}}\bigg|_{t=0} = 3$$

The variance is then $E[Z_1^4] - E[Z_1^2] = 3 - 1 = 2$. Since each variable is independent, $E[Y] = k$ and $\text{Var}(Y) = 2k$

c) By definition and via independence:

$$M_Y(t) = E[e^{tY}] = E\left[\prod_{i=1}^{k} e^{tZ_i^2}\right] = \prod_{i=1}^{k} E[e^{tZ_i^2}] = \prod_{i=1}^{k} \frac{1}{\sqrt{1-2t}} = \frac{1}{(1-2t)^{k/2}}$$

d) To get the third moment, we find the third derivative of the moment generating function:

$$\frac{d}{dt} M_Y(t) = \frac{k}{(1-2t)^{\frac{k+2}{2}}} \implies \frac{d^2}{dt^2} M_Y(t) = \frac{k(k+2)}{(1-2t)^{\frac{k+4}{2}}}$$

This then implies that

$$\frac{d^3}{dt^3} M_Y(t) = \frac{k(k+2)(k+4)}{(1+2t)^{\frac{k+6}{2}}}$$

This evaluated at $t = 0$ is $E[Y^3] = k(k+2)(k+4)$.