MATH 201 HW 2

Written by Nathanael Grand
ngrand@ur.rochester.edu

1

a) The possible outcomes are a sequence of tails (denoted by T) which ends in one heads (denoted by H). Therefore, for all $n \in \mathbb{Z}^+$, define

$$\Omega_n = \{(a_1, a_2, \dots, a_{n-1}, a_n) : a_n = T, a_i = H \text{ for } 1 \leq i < n\}$$

The sample space is then

$$\Omega = \{\infty\} \cup \bigcup_{n=1}^{\infty} \Omega_n$$

Where the set on the left represents the event that an infinite number of heads are produced. Note that there are several ways of defining the sample space. The above is just one method.

b) If $p = 1$, the coin will always be heads, and therefore in every case the flipping will not cease (meaning that the probability of an infinite sequence of heads is 1). If $p = 0$, the only possible outcome is tails, and no sequence of flips will be longer than one toss (meaning that the probability of infinite sequence of heads is 0). Therefore, consider the case where $0 < p < 1$. The probability of getting k heads in a row is the product of the probability of getting one heads multiplied k times:

$$P(\text{k Heads}) = \prod_{i=1}^{k} P(1 \text{ Heads}) = p^k$$

Then, we would like to find the limit as $k \to \infty$. Since $0 < p < 1$, then $p^a < p^b$ when $a > b$. Therefore p^k will be a strictly decreasing sequence of points as k increases. Moreover, $p^{k+1}/p^k = p < 1$ implies that p^k will approach 0 as $k \to \infty$. As k becomes large, the probability that there will be only heads approaches zero. The event where two-face flips an infinite number of heads is a subset of the event where two-face flips n heads (for any $n \in \mathbb{Z}^+$). Therefore, the probability of only flipping heads forever is zero.

c) The following is an alternative method for part (*b*). First, calculate the probability that you see a tails. In other words, we want to find the probability of seeing $(H, H, \dots, H, T)$ of any arbitrary length. This is:

$$P\left(\bigcup_{n=1}^{\infty} \Omega_n\right) = \sum_{n=1}^{\infty} P(\Omega_n) = \sum_{n=1}^{\infty} p^{n-1}(1-p)$$

The probability of the union can be split as each Ω_n are disjoint. If $p = 1$, this will be 0. Therefore $P(\infty) = 1$, and it is certain that there will be infinite heads. Therefore, assume that $0 \leq p < 1$. Computing this geometric series, we get

$$\sum_{n=1}^{\infty} p^{n-1}(1-p) = (1-p)\frac{1}{1-p} = 1$$

By the law of total probability,

$$P\left(\bigcup_{n=1}^{\infty} \Omega_n\right) + P(\infty) = 1 \implies 1 + P(\infty) = 1 \iff P(\infty = 0)$$

In this case, we get that the probability of getting an infinite number of heads is 0

2

Let each person be called a, b and c, and let $A/B/C$ denote the event where person $a/b/c$ loses all games. The probability that at least one person loses all games is the probability of the union $A \cup B \cup C$. To find this probability, we can use the inclusion-exclusion principle:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C)) - P(C \cap A)$$

Note that the intersection $A \cap B \cap C = \emptyset$, since at least one person always wins in each game. Since each person has an equal chance of winning a game, $P(A) = P(B) = P(C)$. It also means that $P(A \cap B) = P(B \cap C) = P(C \cap A)$. The expression above can then be consolidated into:

$$P(A \cup B \cup C) = 3P(A) - 3P(A \cap B)$$

The first term is the probability that only a loses. In each game, the probability that a doesn't win is $\frac{2}{3}$. Therefore, the probability a doesn't win in 6 games is $(\frac{2}{3})^6$. The probability that both a and b lose all games is equivalent to the probability that c wins all games. This is equal to $(\frac{1}{3})^6$. The probability is then

$$P(A \cup B \cup C) = 3\left(\left(\frac{2}{3}\right)^6 - \left(\frac{1}{3}\right)^6\right) = 3\left(\frac{2^6 - 1}{3^6}\right) = \frac{2^6 - 1}{3^5} \approx .26$$

3

a) Let A be the event that the Joker passes the DNA test. Let G be the event where the Joker is guilty. With Bayes' formula,

$$P(X|A) = \frac{P(A|X)P(X)}{P(A|X)P(X) + P(A|X^c)P(X^c)}$$

Since it was given that the culprit will test positive, $P(A|X) = 1$. Therefore we get

$$P(X|A) = \frac{1/200000}{(1)(1/200000) + (1/5000)(199999/200000)} \approx .024$$

b) Let B be the event that the Joker passes Batman's test. We can use Bayes' formula in the same way as above:

$$P(X|B) = \frac{P(B|X)P(X)}{P(B|X)P(X) + P(B|X^c)P(X^c)}$$

Since the perpetrator must have their DNA match, $P(B|X) = 1$, and we get

$$P(X|B) = \frac{1/200000}{(1)(1/200000) + (1/30000)(199999/200000)} \approx .13$$

c) Since we assume the suspect is driving an orange car, we now know the chance that the Joker is the perpetrator is 1/2000. All we do now is apply Bayes' formula with this altered probability. For the original test:

$$P(X|A) = \frac{1/2000}{(1)(1/2000) + (1/5000)(1999/2000)} \approx .714$$

Note that the probability of a false positive remains the same. Doing the same with Batman's test, we get

$$P(X|B) = \frac{1/2000}{(1)(1/2000) + (1/30000)(1999/2000)} \approx .938$$