MATH 201 HW 5

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First,

$$P(X \geq k) = \sum_{n=k}^{\infty} P(X = n)$$

Then,

$$\sum_{k=1}^{\infty} \sum_{n=k}^{\infty} P(X = n)$$

Can be visualized in the following way:

$$\begin{aligned}
= P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + \dots \\
+P(X=2) + P(X=3) + P(X=4) + P(X=5) + \dots \\
+P(X=3) + P(X=4) + P(X=5) + \dots \\
+P(X=4) + P(X=5) + \dots \\
+ P(X=5) + \dots \\
\vdots
\end{aligned}$$

Where each row is the sum $\sum_{n=k}^{\infty} P(X = n)$ (the first row $k = 1$, the second row $k = 2$ and so on). We can notice that there is n copies of $P(X = n)$ in each column. Therefore we can see that

$$\sum_{k=1}^{\infty} P(X \geq k) = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} P(X = n) = \sum_{k=1}^{\infty} kP(X = k)$$

For the geometric series,

$$P(X \geq k) = \sum_{n=k}^{\infty} (1-p)^{n-1} p = (1-p)^{k-1} p \sum_{n=0}^{\infty} (1-p)^n = (1-p)^{k-1} p \left(\frac{1}{1-(1-p)} \right) = (1-p)^{k-1}$$

Then,

$$\sum_{k=1}^{\infty} P(X \geq k) = \sum_{k=1}^{\infty} (1-p)^{k-1} = \frac{1}{1-(1-p)} = \frac{1}{p} = E[X]$$

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The normal distribution $X \sim \mathcal{N}(\mu, \sigma^2)$ has the density function

$$P(X \in A) = \frac{1}{\sigma\sqrt{2\pi}} \int_A e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

If $Z \sim \mathcal{N}(0, 1)$,

$$P(Z \in A) = \frac{1}{\sqrt{2\pi}} \int_A e^{-z^2/2} dz$$

Then,

$$E[Z^3] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^3 e^{-z^2/2} dz$$

Next, we can do integration by parts by letting $u = z^2$, and $dv = ze^{-z^2/2}$. An antiderivative of dv is $v = -e^{-z^2/2}$. Also, $du = 2zdz$. Then,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (z^2)(ze^{-z^2/2}) dz = \frac{1}{\sqrt{2\pi}} (z^2)(-e^{-z^2/2}) \Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2z(-e^{-z^2/2}) dz$$

The term on the left evaluates to 0, as the exponential term dominates z^2 for large $|z|$. We are then left with

$$\frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ze^{-z^2/2} dz = \frac{2}{\sqrt{2\pi}} (-e^{-z^2/2}) \Big|_{-\infty}^{\infty} = 0$$

Another way to see this is that $E[Z^3]$ involves integrating the product of an even and odd function, producing the integral of an odd function over $\mathbb{R}$. Next,

$$E[X^3] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^3 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

We can do a change of variables and let $y = \frac{x-\mu}{\sigma}$. This then yields

$$E[X^3] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma y + \mu)^3 e^{-y^2/2} dy$$

This is close to the integral done previously. We can simplify the cube term to

$$(\sigma y + \mu)^3 = \sigma^3 y^3 + 3\sigma^2 y^2 \mu + 3\sigma y \mu^2 + \mu^3$$

Then, we look at the integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma^3 y^3 + 3\sigma^2 y^2 \mu + 3\sigma y \mu^2 + \mu^3) e^{-y^2/2} dy$$

$$= \frac{\sigma^3}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^3 e^{-y^2/2} dy + \frac{3\sigma^2\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy + \frac{3\sigma\mu^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-y^2/2} dy + \frac{\mu^3}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy$$

The first term disappears, as this is just the integral we calculated previously. The second integral is some constant terms multiplying $E[Z^2] = 1$. The third term is constants multiplying $E[Z] = 0$. The last term will simplify to μ^3. Therefore we get

$$3\sigma^2\mu + \mu^3$$

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This will follow the binomial distribution. Let $p = .00025$, $n = 10000$. The probability that exactly k people win is

$$P(W = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Notice that

$$np(1-p) = (10000)(1/4000)(3999/4000) \approx 2.5 < 10$$

Therefore the exponential distribution should not be accurate. On the other hand

$$np^2 = 10000/(4000^2) \approx 0.00063 < 0.01$$

Therefore an approximation via the Poisson distribution should be more accurate. Lets do this to calculate the probability that the Joker doesn't lose his cool. We can approximate

$$P(W = k) \approx e^{-\lambda} \frac{\lambda^k}{k!}$$

Where $\lambda = E[W] = np = 2.5 = 5/2$. Then,

$$P(W < 3) = P(W = 0) + P(W = 1) + P(W = 2) \approx e^{-5/2} \frac{(5/2)^0}{0!} + e^{-5/2} \frac{(5/2)^1}{1!} + e^{-5/2} \frac{(5/2)^2}{2!}$$

This simplifies to produce the approximation

$$P(W < 3) \approx .54$$