MATH 201 HW 7

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a) The marginal distributions are defined by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

Given that $f(x,y)$ is non-zero for $0 < y < 1$ and $y < x < 2-y$, the domain of $f(x,y)$ is an triangle with vertices $(0,0),(2,0),(1,1)$. However, these vertices are not part of the domain. From this, we get

$$f_X(x) = \int_0^x 3(2-x)ydy = \frac{3(2-x)}{2}y^2\Big|_0^x = \frac{3(2-x)x^2}{2} \quad (0 < x \le 1)$$

$$f_X(x) = \int_0^{2-x} 3(2-x)ydy = \frac{3(2-x)}{2}y^2\Big|_0^{2-x} = \frac{3(2-x)^3}{2} \quad (1 \le x < 2)$$

Therefore we get

$$f_X(x) = \begin{cases} \frac{3(2-x)x^2}{2} & 0 < x \le 1 \\ \frac{3(2-x)^3}{2} & 1 \le x < 2 \\ 0 & \text{else} \end{cases}$$

The marginal distribution f_Y is calculated similarly, although with different integral bounds:

$$f_Y(y) = \int_y^{2-y} 3(2-x)ydx = -3\int_{2-y}^{y} uydu = -3\frac{u^2y}{2}\Big|_{2-y}^{y} = \frac{3(2-y)^2y - 3y^3}{2} \quad (0 < y < 1)$$

Therefore

$$f_Y(y) = \begin{cases} \frac{3(2-y)^2y-3y^3}{2} & 0 < y < 1 \\ 0 & \text{else} \end{cases}$$

b) The formula for expectation is

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = \int_0^1\int_0^x 3xy^2(2-x)dydx + \int_1^2\int_0^{2-x} 3xy^2(2-x)dydx$$

This simplifies to

$$\int_0^1 xy^3(2-x)\big|_0^x dx + \int_1^2 xy^3(2-x)\big|_0^{2-x} dx = \int_0^1 x^4(2-x)dx + \int_1^2 x(2-x)^4dx$$

The first integral evaluates to

$$\frac{2x^5}{5} - \frac{x^6}{6}\Big|_0^1 = \frac{2}{5} - \frac{1}{6} = \frac{7}{30}$$

After a u substitution of $u = 2-x$, the second integral evaluates to

$$-\int_1^0 (2-u)u^4du = \int_0^1 (2-u)u^4du$$

This is the same as the integral above. Therefore we get a total of

$$\frac{7}{30} + \frac{7}{30} = \frac{14}{30} = \frac{7}{15} = E[XY]$$

c) First, $P(X + Y \leq 1) = P(X \leq 1 - Y)$. This can be computed using the joint distribution:

$$P(X \leq 1 - Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{1-y} f(x, y)dxdy = \int_{0}^{1/2} \int_{y}^{1-y} 3(2 - x)ydxdy$$

The upper bound for the integration with respect to y is $1/2$, as the joint distribution is non-zero when $y < x < 2 - y$. If $x > y$, then $y < 1/2$ in order for $X + Y \leq 1$. Evaluating the integral above produces:

$$3 \int_{0}^{1/2} y(2x - x^2/2)\Big|_{y}^{1-y} dy = 3 \int_{0}^{1/2} y(2(1 - y) - (1 - y)^2/2 - 2y + y^2/2)dy$$

This simplifies to

$$3 \int_{0}^{1/2} y(2 - 2y - (1 - 2y + y^2)/2 - 2y + y^2/2)dy$$

Which becomes

$$3 \int_{0}^{1/2} y(3/2 - 3y)dy = 3 \left(\frac{3y^2}{4} - y^3 \right) \Big|_{0}^{1/2} = 3 \left(\frac{3}{16} - \frac{1}{8} \right) = \frac{3}{16}$$

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The trinomial coefficient counts the number of ways one can partition n into three sets. A specific ordering of W lectures with white chalk, Y lectures with yellow chalk and G lectures with green chalk occurs with a probability of $p_1^W p_2^Y p_3^G$ (Independence is used here). The trinomial coefficient counts the number of sequences of lectures for which W, Y, G white yellow and green chalks are used. Therefore

$$p(W, Y, G) = \binom{n}{WYG} p_1^W p_2^Y p_3^G = \frac{n!}{W!Y!G!} p_1^W p_2^Y p_3^G$$

This idea extends to an arbitrary selection of chalk colors. If there is also red chalk,

$$P(W, Y, G, R) = \binom{n}{WYGR} p_1^W p_2^Y p_3^G p_4^R = \frac{n!}{W!Y!G!R!} p_1^W p_2^Y p_3^G p_4^R$$