MATH 201 HW 7

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a) Let I_j be the random variable equal to 1 if any students step out at the j^{th} floor and 0 otherwise. Then, $N = \sum_{j=1}^{11} I_j$. The probability that nobody leaves the elevator on the j^{th} floor is

$$P(I_j = 0) = \left(\frac{10}{11}\right)^7$$

This comes from the uniformity of the floor selection as well as the independence of each person. This means that

$$P(I_j = 1) = 1 - \frac{10^7}{11^7}$$

Therefore,

$$E[I_j] = 0\left(\frac{10}{11}\right)^7 + 1\left(1 - \frac{10^7}{11^7}\right) = 1 - \frac{10^7}{11^7}$$

By the linearity of expectation

$$E[N] = E\left[\sum_{j=1}^{11} I_j\right] = \sum_{j=1}^{11} E[I_j] = 11 - \frac{10^7}{11^6} = \frac{11^7 - 10^7}{11^6} \approx 5.4$$

b) The random variable X_j follows a binomial distribution with parameters $n = 7$, $p = 1/11$. Therefore,

$$P(X_j = k) = \binom{7}{k}\left(\frac{1}{11}\right)^k\left(\frac{10}{11}\right)^{7-k} = \binom{7}{k}\frac{10^{7-k}}{11^7} = \frac{7!}{k!(7-k)!}\frac{10^{7-k}}{11^7}$$

The expectation of a random variable following the binomial distribution is np. Therefore

$$E[X_j] = \frac{7}{11}$$

c) The covariance between X_1 and X_2 is

$$\text{Cov}(X_1, X_2) = E[X_1X_2] - E[X_1]E[X_2]$$

Let the seven people be labeled $\{1, 2, \ldots, 7\}$. Let J_i be a random random variable which is 1 if the i^{th} person steps out of the elevator on the first floor, and 0 otherwise. Define K_n in the same way for the second floor. Then,

$$X_1 = \sum_{i=1}^{7} J_i, \quad X_2 = \sum_{n=1}^{7} K_n \implies X_1X_2 = \sum_{i=1}^{7}\sum_{n=1}^{7} J_iK_n$$

Expectation is linear, so we just need to figure out $E[J_iK_n]$. If $i = n$, the only outcome is 0, as one person can only get out on one floor. Therefore this sum can be simplified to be over $i \neq n$. Then,

$$E[J_iK_n] = (1)P(J_i = 1, K_n = 1) + (0)P(J_i = 0 \text{ or } K_n = 0) = P(J_i = 1, K_n = 1)$$

Since each person is independent, this is just the product

$$P(J_i = 1)P(K_n = 1) = \frac{1}{11}\frac{1}{11} = \frac{1}{121}$$

If $j \neq n$, there are $7^2 - 7 = 42$ terms. Therefore

$$E[X_1X_2] = \frac{42}{121}$$

The covariance is then

$$\frac{42}{121} - \frac{7}{11}\frac{7}{11} = \frac{42-49}{121} = -\frac{7}{121}$$

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We can write

$$X = \sum_{i=1}^{20} Y_i$$

Where Y_i is the random variable which is 1 if the i^{th} student is American, and 0 otherwise. First, it is the case, that $P(Y_i) = \frac{4}{5}$ for all i. This follows from the exchangability of sampling without replacement. With this fact,

$$E[X] = \sum_{i=1}^{20} E[Y_i] = \sum_{i=1}^{20} P(Y_i = 1) = 20P(Y_i = 1) = \frac{20(4)}{5} = \frac{80}{5} = 16$$

For the variance, we use the fact that X is described by the Hypergeometric distribution. This has a variance of

$$\frac{N-n}{N-1}npq$$

Where $N = 40$ is the total number of people, $n = 20$ is the number of trials, $p = \frac{4}{5}$ is the ratio between Americans and Non-Americans, and $q = 1 - p$. Therefore

$$\text{Var}(X) = \frac{30}{49}\frac{20(4)}{5^2} = \frac{(6)(16)}{49} = \frac{96}{49}$$

For more information on the derivation for this, check examples 8.7 and 8.30 from the textbook.

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The covariance is given by

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

First,

$$E[X] = \int_{-\infty}^{\infty} p_X(x)xdx = 0$$

$$E[XY] = \int_{-\infty}^{\infty} p_X(x)xf(x)dx = 0$$

As $p_X(x)x$ and $p_X(x)xf(x)$ are odd functions. This is the case since $p_X(x)$ and $f(x)$ are even, while x is odd. The product of even and odd functions are odd. Therefore the covariance is 0 provided that $E[Y]$ is finite. Suppose that $f(x) \neq c$, where c is a constant. Then, X and Y are not independent, as information about X completely determines Y. If f is constant, then knowledge about X does not influence the output of f, and they are independent.