MATH 201 HW 9

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a) The moment generating function is

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{n} e^{tk} P(e^{tX} = e^{tk}) = \sum_{k=0}^{n} e^{tk} P(X = k) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$

This can be rewritten as

$$M_X(t) = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k} = (pe^t + 1 - p)^n = (pe^t - p + 1)^n$$

b) Moments can be found by taking derivatives of the moment generating function and evaluating those derivatives at $t = 0$:

$$E[X^3] = \frac{d^3}{dt^3} M_X(t)\bigg|_{t=0}$$

First,

$$\frac{d}{dx} M_X(t) = n(pe^t - p + 1)^{n-1} pe^t$$

$$\implies \frac{d^2}{dx^2} M_X(t) = (n)(n-1)(pe^t - p + 1)^{n-2} p^2 e^{2t} + n(pe^t - p + 1)^{n-1} pe^t$$

$$\implies \frac{d^3}{dx^3} M_X(t) = (n)(n-1)(n-2)(pe^t - p + 1)^{n-3} p^3 e^{3t} + 2(n)(n-1)(pe^t - p + 1)^{n-2} p^2 e^{2t}$$
$$+ n(n-1)(pe^t - p + 1)^{n-1} p^2 e^{t^2} + n(pe^t - p + 1)^{n-1} pe^t$$

This evaluated at $t = 0$ is

$$E[X^3] = (n)(n-1)(n-2)p^3 + 2(n)(n-1)p^2 + (n)(n-1)p^2 + np$$
$$= (n)(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

2

a) Markov's inequality gives us

$$P(X \geq 16) \leq \frac{E[X]}{16} = \frac{10}{16} = \frac{5}{8}$$

b) Chebyshev's Inequality inequality gives us

$$P(|X - 10| \geq 6) \leq \frac{\mathrm{Var}(X)}{6^2}$$

Then, $\{X - 10 \geq 6\} \subseteq \{|X - 10| \geq 6\}$ implies that

$$P(X \geq 16) = P(X - 10 \geq 6) \leq P(|X - 10| \geq 6) \leq \frac{\mathrm{Var}(X)}{6^2} = \frac{3}{6^2} = \frac{1}{12}$$

c) Define the random variable

$$S = \sum_{i=1}^{300} Y_i$$

Since each Y_i and Y_j are identical and independent,

$$E[S] = \sum_{i=1}^{300} E[Y_i] = 300(10) = 3000, \quad \text{Var}(S) = \sum_{i=1}^{300} \text{Var}(Y_i) = 300(3) = 900$$

We can then use Chebyshev's inequality to estimate:

$$P(S \geq 3031) \leq P(|S - 3000| \geq 31) \leq \frac{\text{Var(S)}}{31^2} = \frac{900}{961} \approx .937$$

We can also try with Markov's:

$$P(S \geq 3031) \leq \frac{E[S]}{3031} = \frac{3000}{3031} \approx .99$$

Again we see that Chebyshev's inequality is sharper. Sharper than both will be the bound we get via the CLT. Let $\mu = 3000$, $\sigma^2 = 900$

$$P\left(\frac{S-\mu}{\sigma} \leq z\right) \approx \Phi(z)$$

Therefore

$$1 - \Phi(z) \approx P\left(\frac{S-\mu}{\sigma} > z\right) = P\left(S > z\sigma + \mu\right)$$

Since we want $S > 3030$, we solve for $z\sigma + \mu = 3030$,

$$z(30) + 3000 = 3030 \implies z = 1$$

Therefore we want to find $1 - \Phi(1) \approx 1 - 0.8413 = 0.1587$.

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Let X_i be the random variable which describes the time it takes Nate to eat the i^{th} hotdog in the competition. What we want to inspect is

$$S = \sum_{i=1}^{64} X_i$$

The random variable S describes the time that it takes for Nate to eat 64 hotdogs. Since it is the sum of normal random variables, S is also normally distributed with

$$E[S] = 64(15) = 960, \quad \text{Var}(S) = 64(16) = 1024$$

Then, if we let $\mu = E[S]$, $\sigma^2 = \text{Var}(S)$, the CLT gives us

$$P\left(\frac{S-\mu}{\sigma} \leq z\right) \approx \Phi(z)$$

We then want to solve for z. What we will look for is

$$P\left(S \leq z\sigma + \mu\right)$$

Since we want $S \leq (15)(60) = 900$ (converting minutes to seconds), we solve for

$$z\sigma + \mu = 900 \implies z = \frac{900 - \mu}{\sigma} = \frac{900 - 960}{32} = -\frac{60}{32} = -\frac{30}{16} = -1.875$$

Therefore we can estimate this probability as

$$\Phi(-1.875) \approx .0301$$