Math 201

Final

December 16, 2014

Course ID number: ____________

Circle your professor's name: **Bridy** **Mkrtchyan** **Mueller**

- **Both parts A and B count towards your score on the final. In addition, your score on part A can replace a bad midterm grade.**
- **No calculators are allowed on this exam, but you are allowed two sheets of paper with writing on both sides.**
- **You need not reduce such expressions as** $\binom{100}{30}$ **and** $50!$ **to a number.**

Part A		
QUESTION	VALUE	SCORE
1	12	
2	13	
3	12	
4	13	
5	12	
6	13	
7	12	
8	13	
TOTAL	100	

Part B		
QUESTION	VALUE	SCORE
9	12	
10	12	
11	13	
12	13	
13	13	
14	12	
15	12	
16	13	
TOTAL	100	

Part A

1. (12 points) How many "words" can we form by rearranging the letters in "Hillary Clinton"? For the purposes of this problem, a word is a sequence of letters in a row. The blank space in "Hillary Clinton" is not included as a letter.

2. (13 points) Suppose your performance on this final depends on the phase of the moon on the evening you were born. If it was a full moon, you will pass with probability 99/100. If it was a half moon, you will pass with probability 9/10. If it was a new moon, you will pass with probability 8/10. Assume that there are no other possibilities for the moon, and that originally all three of these outcomes were equally likely. Furthermore, neither you nor your parents remember the phase of the moon on the evening you were born. Assuming you passed, find the probability of a full moon on the evening you were born.

3. (12 points) One of our collaborators claims that every mathematical publication has mistakes. In 1814 Marquis de Laplace, one of the founders of probability, published "A Philosophical Essay on Probabilities" (in French). The English translation has 305 pages. Suppose each page has a probability of 7% of having a mistake. Assume that the events of mistakes on different pages are independent, and that the probability of more than one mistake on a page is very small and can be ignored.

(a) What is the expected number of mistakes in the book?

(b) What is the probability of exactly 10 mistakes in the book?

(c) Use the Poisson distribution to give an approximate answer for part (b).

4. (13 points) A class has two exams. You know that 80% of the students passed the first exam and that 40% of the students did not pass the second exam. If a student passed the first exam, then the probability that they also passed the second exam is $\frac{5}{8}$.

a) What is the probability that a randomly selected student passed both exams?

b) What is the probability that a randomly selected student did not pass either exam?

5. (12 points) Let X be a random variable that only takes the values 0, 1, 2. Let $p(x)$ be the probability mass function of X. Suppose that you know that $E[X] = 1$ and $\text{Var}(X) = \frac{1}{2}$. Find $p(0)$, $p(1)$, and $p(2)$.

6. (13 points) A sack has $2n$ socks. $2n - 2$ of these are white and the remaining two are black. Suppose n people randomly choose a pair of socks each.

(a) What is the probability that 3 or more people have mismatched socks?

(b) What is the probability that there is someone who has mismatched socks? You do not need to simplify your final answer.

Hint: Use the inclusion-exclusion formula.

7. (12 points) Math 201 has three sections which have respectively $35, 50$ and 65 students.

(a) Suppose we randomly choose, with equal probability, one of the three instructors and X is the size of the class the instructor is teaching. Compute $E[X]$.

(b) Suppose we randomly choose, with equal probability, one of the 150 students and Y is the size of the class the student is taking. Compute $E[Y]$. You do not need to simplify your answer.

8. (13 points) Let A be the area of a square whose side length is chosen uniformly at random from the interval $(0, 2)$.

(a) Compute the expected value of A.

(b) Compute the variance of A.

Part B

9. (12 points) Recall that $\Phi(x) = P(Z \leq x)$ where Z is a standard normal random variable. That is, Φ is the cumulative distribution function of the standard normal. Suppose that Y has a normal distribution with mean 5 and variance 4. Find $P(3 < Y < 4)$ in terms of Φ.

10. (12 points) (a) Let Y_1, Y_2 be independent binomial random variables with parameters (n_1, p) and (n_2, p) respectively. Note that p is the same for both random variables. Let $X = Y_1 + Y_2$. Explain why X is binomial with parameters $(n_1 + n_2, p)$.

Hint: Think of how a binomial random variable arises from Bernoulli random variables.

(b) Assuming the result of part (a), find the conditional probability mass function $p_{Y_1|X}(y|x)$.

11. (13 points) Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. (independent identically distributed) random variables with mean 0 and variance 1. Let $S_n = X_1 + \cdots + X_n$, and recall that $\mathrm{Var}(S_n) = n$ in this situation. Use Chebyshev's inequality to prove that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n^{3/4}}\right| > \varepsilon\right) = 0.$$

12. (13 points) Let X and Y be independent random variables that are both uniformly distributed on the interval $(0, 3)$. Compute $P\{X + Y > 4\}$.

13. (13 points) Let X and Y be random variables with joint density equal to

$$f(x,y) = \begin{cases} 6e^{-2x-3y} & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\mathrm{Cov}(X,Y)$.

14. (12 points) Let X be an exponential random variable with parameter λ. Compute the density of X^2.

15. (12 points) In a four person relay race each person completes one lap of the course and the time assigned to the team is the sum of the four individual times. Suppose the time each person takes to run their section is a random variable with probability density function

$$f(x) = \begin{cases} -6x^2 + 18x - 12 & \text{if } 1 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected time that the four person team will take to run the course?

16. (13 points) A person found a great sale on bulbs for a particular lamp and bought 100 of them. Assuming that the lifetime of a bulb is an exponentially distributed random variable with expectation 1 year, use the normal approximation to estimate the probability that the bulbs will last at least 75 years. You can leave your answer in terms of the c.d.f of a standard normal random variable.

Hint: For the exponential distribution, the variance is the square of the mean.