Department of Mathematics
University of Rochester

MTH 201
Spring 2015

Final Exam, May 5, 2015

Examiner: G. Richards
Duration: 180 minutes

Total: 100 points

Family Name: ______________________________
(Please Print)

Given Name(s): ______________________________
(Please Print)

Student ID Number: ______________________________

FOR MARKER'S USE ONLY			
Problem 1:	/10	Problem 2:	/10
Problem 3:	/10	Problem 4:	/10
Problem 5:	/10	Problem 6:	/10
Problem 7:	/10	Problem 8:	/10
Problem 9:	/10	Problem 10:	/10
		TOTAL:	/100

1. (10 points) Let S be the set of numbers $S = \{1, 2, \ldots, 14, 15\}$.

(a) How many subset of S (without order) contain exactly 4 numbers?

(b) How many subsets of S with size 4 contain at least one of the numbers $1, 2, 3, 4, 5$?

2. (10 points) A standard deck of 52 playing cards contains exactly 4 “Ace” cards. Suppose a (shuffled) deck of cards a dealt out (i.e. each card is revealed, one after another).

(a) What is the probability that the 8th card dealt out is an Ace?

(b) What is the probability that the *first* Ace to be drawn occurs on the 8th card?

3. (10 points) Suppose there is a 70% chance that event A will occur. If A does not occur, then there is a 20% chance that B will occur. What is the probability that at least one of the events A or B will occur?

4. (10 points) Suppose that the probability that an item produced by a certain machine is defective is 0.05. Suppose that 100 of these items are produced by this machine.

(a) What is the expected number of defective items?

(b) What is the probability that at most one of these 100 items is defective?

(c) Use the Poisson distribution to approximate the probability that at most 4 of these 100 items is defective.

5. (10 points) Suppose that, for some constant $c > 0$, and integer $n > 0$, the random variable X has probability density function $f(x) = \begin{cases} cx^n & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

(a) Find the constant $c > 0$.

(b) For fixed $a \in (0, 1)$, find $P(X > a)$.

(c) For fixed $a \in (0, 1)$, find $\lim_{n\to\infty} P(X > a)$.

6. (10 points) Suppose X is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 9$. Find an approximation for $P(X > 0)$, and express your answer in terms of $\Phi(x)$, the cumulative distribution function of the standard normal random variable.

7. (10 points) Let $f(x,y) = \begin{cases} 24xy & \text{if } 0 < x < 1,\ 0 < y < 1, \text{ and } 0 < x+y < 1, \\ 0 & \text{otherwise.} \end{cases}$

(a) Show that $f(x,y)$ is a joint probability density function for some continuous random variables X and Y.

(b) Are X and Y independent? Justify your answer.

8. (10 points) Let $M_{X+Y}(t)$, $M_X(t)$ and $M_Y(t)$ denote the moment generating functions of random variables $X+Y$, X and Y, respectively.

(a) Suppose that X and Y are independent. Show that $M_{X+Y}(t) = M_X(t)M_Y(t)$.

(b) Now suppose that X and Y are independent Poisson random variables with parameters λ_1 and λ_2, respectively. Show that the moment generating function $M_{X+Y}(t)$ of $X+Y$ is given by $M_{X+Y}(t) = \exp\{(\lambda_1 + \lambda_2)(e^t - 1)\}$.

NOTE: Here we have used the notation $\exp\{x\} = e^x$ to avoid too many superscripts.

HINT: You should begin by deriving the moment generating functions of X and Y individually.

(c) It turns out that the moment generating function of a random variable uniquely determines the distribution of that random variable. That is, if $M_{X_1}(t) = M_{X_2}(t)$ for all sufficiently small $t \in \mathbb{R}$, then the random variables X_1 and X_2 have the same distribution.

Based on this fact, if X and Y are independent Poisson random variables with parameters λ_1 and λ_2, respectively, what can you conclude about the distribution of $X + Y$?

9. (10 points) Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables. Let $\overline{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$ denote the corresponding sample mean.

In this setting, for each $i = 1, 2, \ldots, n$, the random variables $X_i - \overline{X}$ are called the *deviations* from the sample mean.

(a) Show that $\text{Cov}(X_i - \overline{X}, \overline{X}) = 0$ for each $i = 1, 2, \ldots, n$.

(b) Is the sample mean $\overline{X}$ is always independent of the deviation $X_i - \overline{X}$ for each fixed $i = 1, 2, \ldots, n$?

10. (10 points) For both of these problems, you may express your answer in terms of the cumulative distribution function $\Phi(x)$ of the standard normal random variable.

(a) Suppose 10 fair 6-sided dice are rolled. Approximate the probability that the sum of the 10 rolls lies between 30 and 40.

(b) Now suppose 100 fair 6-sided dice are rolled. Let X_i denote the outcome of the ith roll. For fixed a with $1 < a < 6$, find an approximation for $P\left(X_1 \cdot X_2 \cdots X_{99} \cdot X_{100} \leq a^{100}\right)$