Math 201

Final ANSWERS

December 17, 2014

1. (12 points) How many "words" can we form by rearranging the letters in "Hillary Clinton"? For the purposes of this problem, a word is a sequence of letters in a row. The blank space in "Hillary Clinton" is not included as a letter.

Answer:

There are 14 letters in "Hillary Clinton", leading to 14! arrangements. However, some of the letters are duplicated. Since there are 3 l's, we should divide by 3!. Likewise, there are 2 i's and 2 n's, so we should also divide twice by 2!. The final answer would be:

$$\frac{14!}{3! \cdot (2!)^2}$$

2. (13 points) Suppose your performance on this final depends on the phase of the moon on the evening you were born. If it was a full moon, you will pass with probability 99/100. If it was a half moon, you will pass with probability 9/10. If it was a new moon, you will pass with probability 8/10. Assume that there are no other possibilities for the moon, and that originally all three of these outcomes were equally likely. Furthermore, neither you nor your parents remember the phase of the moon on the evening you were born. Assuming you passed, find the probability of a full moon on the evening you were born.

Answer:

Let F, H, N be the events of a full, half, or new moon on the evening you were born. Let S be the event that you pass. We want to find $P(F|S)$. We know that

$$P(S|F) = \frac{99}{100} \qquad P(S|H) = \frac{9}{10} \qquad P(S|N) = \frac{8}{10}$$

and

$$P(F) = P(H) = P(N) = \frac{1}{3}$$

By Bayes' formula,

$$P(F|S) = \frac{P(S|F)P(F)}{P(S)}$$

We need to find $P(S)$. But,

$$\begin{aligned} P(S) &= P(S \cap F) + P(S \cap H) + P(S \cap N) \\ &= P(S|F)P(F) + P(S|H)P(H) + P(S|N)P(N) \\ &= \frac{99}{100} \cdot \frac{1}{3} + \frac{9}{10} \cdot \frac{1}{3} + \frac{8}{10} \cdot \frac{1}{3} \\ &= \frac{269}{300} \end{aligned}$$

Finally,

$$P(F|S) = \frac{P(S|F)P(F)}{P(S)} = \frac{\frac{99}{100} \cdot \frac{1}{3}}{\frac{269}{300}} = \frac{99}{269}$$

3. (12 points) One of our collaborators claims that every mathematical publication has mistakes. In 1814 Marquis de Laplace, one of the founders of probability, published "A Philosophical Essay on Probabilities" (in French). The English translation has 305 pages. Suppose each page has a probability of 7% of having a mistake. Assume that the events of mistakes on different pages are independent, and that the probability of more than one mistake on a page is very small and can be ignored.

(a) What is the expected number of mistakes in the book?

(b) What is the probability of exactly 10 mistakes in the book?

(c) Use the Poisson distribution to give an approximate answer for part (b).

Answer:

(a) The number of mistakes X_k on the kth page is a Bernoulli variable with $p = \frac{7}{100}$, and therefore $EX_k = p = \frac{7}{100}$. Since there are 305 pages, the expectation of the total number of mistakes is

$$E\left[X_1 + \cdots + X_{305}\right] = 305 \cdot E[X_1] = \frac{305 \cdot 7}{100} = 21.35$$

(b) Since the total number N of mistakes is the sum of independent Bernoulli variables, it has the binomial distribution with $n = 305$ and $p = \frac{7}{100}$. therefore

$$P(N = 10) = \binom{n}{10} p^{10} q^{n-10} = \binom{305}{10} \left(\frac{7}{100}\right)^{10} \left(\frac{93}{100}\right)^{295}$$

(c) Recall that if N is binomial (n, p), and p is small and n is large, then N is approximately Poisson with parameter $\lambda = pn$. In our case $\lambda = pn = 305 \cdot \frac{7}{100} = 21.35$. Thus

$$P(N = 10) \approx \frac{\lambda^{10}}{10!} e^{-\lambda} = \frac{(21.35)^{10}}{10!} e^{-21.35}.$$

4. (13 points) A class has two exams. You know that 80% of the students passed the first exam and that 40% of the students did not pass the second exam. If a student passed the first exam, then the probability that they also passed the second exam is $\frac{5}{8}$.

a) What is the probability that a randomly selected student passed both exams?

b) What is the probability that a randomly selected student did not pass either exam?

Answer:

Let E be the event of passing exam 1 and F be the event of passing exam 2. For a), we use the multiplication formula to get

$$P(EF) = P(F|E)P(E) = .5.$$

For b), we compute

$$P(E \cup F) = P(E) + P(F) - P(EF) = .8 + .6 - .5 = .9$$

and then $P(E^C F^C) = 1 - P(E \cup F) = .1$.

5. (12 points) Let X be a random variable that only takes the values 0, 1, 2. Let $p(x)$ be the probability mass function of X. Suppose that you know that $E[X] = 1$ and $\mathrm{Var}(X) = \frac{1}{2}$. Find $p(0)$, $p(1)$, and $p(2)$.

Answer:

Let $a = p(0)$, $b = p(1)$, and $c = p(2)$. We must have $a + b + c = 1$. From the formula for expected value,

$$0a + 1b + 2c = 1$$

and from the usual formula for computing variance,

$$0^2a + 1^2b + 2^2c - 1^2 = \frac{1}{2}.$$

Solving these equations yields $a = \frac{1}{4}$, $b = \frac{1}{2}$, and $c = \frac{1}{4}$.

6. (13 points) A sack has $2n$ socks. $2n - 2$ of these are white and the remaining two are black. Suppose n people randomly choose a pair of socks each.

(a) What is the probability that 3 or more people have mismatched socks?

(b) What is the probability that there is someone who has mismatched socks? You do not need to simplify your final answer.

Hint: Use the inclusion-exclusion formula.

Answer:

(a) Since every person who picks mismatched socks must have a black sock, and there are only two black socks, the probability that 3 or more people have mismatched socks is zero.

(b) Let A_i be the event that person i has mismatched socks. We are looking for $a = P(\cup_{i=1}^n A_i)$. We will use the inclusion-exclusion formula to compute this probability. Since by the first part it is impossible for more than 2 people to have mismatches socks, i.e. $P(A_{i_1} \cap \cdots \cap A_{i_r}) = 0$ if $r > 2$ and $i_1, \ldots, i_r$ are distinct, we have

$$a = \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j).$$

By symmetry we have

$$a = nP(A_1) - \binom{n}{2} P(A_1 \cap A_2).$$

There are $\binom{2n}{2}$ ways to choose a pair of socks for the first person and $2(2n-2)$ ways so that they do not match (choose one of each color), thus

$$P(A_1) = \frac{2(2n-2)}{\binom{2n}{2}}.$$

Similarly, there are $\binom{2n}{4}\binom{4}{2}$ ways to choose the socks for person 1 and person 2, and in only $2(2n-2)(2n-3)$ of those both of them get mismatched socks. Thus

$$P(A_1 \cap A_2) = \frac{2(2n-2)(2n-3)}{\binom{2n}{4}\binom{4}{2}}.$$

Combining these we get

$$a = n\frac{2(2n-2)}{\binom{2n}{2}} - \binom{n}{2}\frac{2(2n-2)(2n-3)}{\binom{2n}{4}\binom{4}{2}}.$$

7. (12 points) Math 201 has three sections which have respectively $35, 50$ and 65 students.

(a) Suppose we randomly choose, with equal probability, one of the three instructors and X is the size of the class the instructor is teaching. Compute $E[X]$.

(b) Suppose we randomly choose, with equal probability, one of the 150 students and Y is the size of the class the student is taking. Compute $E[Y]$. You do not need to simplify your answer.

Answer:

(a) X is a random variable which takes the values $40, 50$ and 70 with probability $1/3$ each, thus $E(X) = 35 \cdot \frac{1}{3} + 50 \cdot \frac{1}{3} + 65 \cdot \frac{1}{3} = 50$.

(b) Y is a random variable which is 35 with probability $35/150$, 50 with probability $50/150$ and 65 with probability $65/150$. Thus $E(Y) = 35 \cdot \frac{35}{150} + 50 \cdot \frac{50}{150} + 65 \cdot \frac{65}{150}$.

8. (13 points) Let A be the area of a square whose side length is chosen uniformly at random from the interval $(0, 2)$.

(a) Compute the expected value of A.

(b) Compute the variance of A.

Answer:

(a) Let X be the side length of the square. It has density $f(t) = 1/2$ if $t \in (0, 2)$ and 0 otherwise. Hence

$$E(A) = E(X^2) = \int_0^2 x^2 \frac{1}{2} dx = \frac{x^3}{6} |_0^2 = \frac{4}{3}.$$

(b) To compute the variance of A, let's compute $E(A^2)$ first. We have

$$E(A^2) = E(X^4) = \int_0^2 x^4 \frac{1}{2} dx = \frac{x^5}{10} |_0^2 = \frac{16}{5}.$$

Thus

$$Var(A) = E(A^2) - E(A)^2 = \frac{16}{5} - \frac{16}{9} = \frac{64}{45}.$$

9. (12 points) Recall that $\Phi(x) = P(Z \leq x)$ where Z is a standard normal random variable. That is, Φ is the cumulative distribution function of the standard normal. Suppose that Y has a normal distribution with mean 5 and variance 4. Find $P(3 < Y < 4)$ in terms of Φ.

Answer:

If Y is a normal random variable with mean μ and variance σ^2, then

$$Z = \frac{Y - \mu}{\sigma}$$

has the standard normal distribution. In our case $\mu = 5$ and $\sigma = \sqrt{4} = 2$. Thus, by subtracting 5 and dividing by 2, we find

$$\begin{aligned} P(3 < Y < 4) &= P(-2 < Y - \mu < -1) \\ &= P\left(-1 < \frac{Y-\mu}{\sigma} < -\frac{1}{2}\right) \\ &= P\left(-1 < Z < -\frac{1}{2}\right) \\ &= P\left(Z < -\frac{1}{2}\right) - P(Z < -1) \\ &= \Phi\left(-\frac{1}{2}\right) - \Phi(-1) \end{aligned}$$

If we want, we could express the answer using positive numbers in our function Φ:

$$\begin{aligned} P(3 < Y < 4) &= \Phi\left(-\frac{1}{2}\right) - \Phi(-1) \\ &= \left[1 - \Phi\left(\frac{1}{2}\right)\right] - [1 - \Phi(1)] \\ &= \Phi(1) - \Phi\left(\frac{1}{2}\right). \end{aligned}$$

10. (12 points) (a) Let Y_1, Y_2 be independent binomial random variables with parameters (n_1, p) and (n_2, p) respectively. Note that p is the same for both random variables. Let $X = Y_1 + Y_2$. Explain why X is binomial with parameters $(n_1 + n_2, p)$.

Hint: Think of how a binomial random variable arises from Bernoulli random variables.

(b) Assuming the result of part (a), find the conditional probability mass function $p_{Y_1|X}(y|x)$.

Answer:

(a) Recall that Y_1 is a sum of n_1 independent Bernoulli variables with parameter p, and Y_2 is a sum of n_2 such independent variables. Since Y_1 and Y_2 are independent, the sum $X = Y_1 + Y_2$ is a sum of $n_1 + n_2$ independent Bernoulli random variables with parameter p. Thus X is binomial with parameters $(n_1 + n_2, p)$.

(b) The probability mass function is defined to be

$$
\begin{aligned}
p_{Y_1|X}(y|x) &= P(Y_1 = y|X = x) \\
&= \frac{P(Y_1 = y, X = x)}{P(X = x)} \\
&= \frac{P(Y_1 = y, Y_2 = x - y)}{P(X = x)} \\
&= \frac{P(Y_1 = y)P(Y_2 = x - y)}{P(X = x)} \\
&= \frac{\binom{n_1}{y} p^y q^{n_1 - y} \binom{n_2}{x-y} p^{x-y} q^{n_2 - x + y}}{\binom{n_1+n_2}{x} p^x q^{n_1+n_2-x}} \\
&= \frac{\binom{n_1}{y}\binom{n_2}{x-y}}{\binom{n_1+n_2}{x}}
\end{aligned}
$$

11. (13 points) Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. (independent identically distributed) random variables with mean 0 and variance 1. Let $S_n = X_1 + \cdots + X_n$, and recall that $\mathrm{Var}(S_n) = n$ in this situation. Use Chebyshev's inequality to prove that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n^{3/4}}\right| > \varepsilon\right) = 0.$$

Answer:

Using Chebyshev's inequality, and assuming $\varepsilon > 0$, we get

$$
\begin{aligned}
P\left(\left|\frac{S_n}{n^{3/4}}\right| > \varepsilon\right) &= P\left(|S_n| > n^{3/4}\varepsilon\right) \\
&\le \frac{\mathrm{Var}(S_n)}{n^{3/2}\varepsilon^2} \\
&= \frac{n}{n^{3/2}\varepsilon^2} \\
&= \frac{1}{n^{1/2}\varepsilon^2} \\
&\to 0 \qquad \text{as } n \to \infty.
\end{aligned}
$$

12. (13 points) Let X and Y be independent random variables that are both uniformly distributed on the interval $(0, 3)$. Compute $P\{X + Y > 4\}$.

Answer:

The joint density of X and Y is

$$f(x,y) = \begin{cases} \frac{1}{9} & \text{if } 0 < x < 3 \text{ and } 0 < y < 3 \\ 0 & \text{otherwise} \end{cases}$$

To find this probability, we integrate over the region where $X + Y > 4$, which means that $Y > 4 - X$, and certainly also $Y < 3$, so

$$4 - X < Y < 3.$$

Therefore $1 < X < 3$ (note that X cannot be less than 1, because then X and Y could not possibly sum to 4.) So the probability is

$$\int_1^3 \int_{4-x}^3 \frac{1}{9}\, dy\, dx = \frac{1}{9}\int_1^3 (3 - (4-x))\, dx = \frac{1}{9}\int_1^3 (x-1)\, dx = \frac{2}{9}.$$

13. (13 points) Let X and Y be random variables with joint density equal to

$$f(x,y) = \begin{cases} 6e^{-2x-3y} & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\text{Cov}(X, Y)$.

Answer:

This can be done by several integrations, but much easier is the following: X and Y are independent because their joint density factors into a function of X times a function of Y. So their covariance is zero.

14. (12 points) Let X be an exponential random variable with parameter λ. Compute the density of X^2.

Answer:

First, let's compute the c.d.f. of X^2. For $a > 0$ we have

$$F_{X^2}(a) = P(X^2 \le a) = P(X \le \sqrt{a}) = 1 - e^{-\lambda\sqrt{a}}.$$

Thus the density of X is

$$f_{X^2}(a) = \frac{d}{da}F_{X^2}(a) = \frac{\lambda}{2\sqrt{a}}e^{-\lambda\sqrt{a}}$$

if $a > 0$ and $f_{X^2}(a) = 0$ if $a \le 0$.

15. (12 points) In a four person relay race each person completes one lap of the course and the time assigned to the team is the sum of the four individual times. Suppose the time each person takes to run their section is a random variable with probability density function

$$f(x) = \begin{cases} -6x^2 + 18x - 12 & \text{if } 1 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected time that the four person team will take to run the course?

Answer:

Let X_i be the time it takes person i to run their section. We are interested in $a = E(X_1 + \cdots + X_4)$ which by linearity of expectation is $E(X_1) + \cdots + E(X_4)$. We have

$$\begin{aligned} E(X_i) &= \int_1^2 x(-6x^2 + 18x - 12)dx \\ &= (-\frac{6}{4}x^4 + 6x^3 - 6x^2)|_1^2 \\ &= (-24 + 48 - 24) - (-3/2 + 6 - 6) = 3/2, \end{aligned}$$

which implies $a = 4 \cdot 3/2 = 6$.

16. (13 points) A person found a great sale on bulbs for a particular lamp and bought 100 of them. Assuming that the lifetime of a bulb is an exponentially distributed random variable with expectation 1 year, use the normal approximation to estimate the probability that the bulbs will last at least 75 years. You can leave your answer in terms of the c.d.f of a standard normal random variable.

Hint: For the exponential distribution, the variance is the square of the mean.

Answer:

Let X_i be the lifetime of the i'th bulb. Then the bulbs will last $S_{100} = X_1 + \cdots + X_{100}$ years. Since X_i is exponential with mean 1, it also has variance 1. Hence by the Central Limit Theorem we have

$$P(\frac{S_{100} - 100 \cdot 1}{\sqrt{100 \cdot 1}} \geq t) \approx 1 - \Phi(t),$$

where $\Phi(t)$ is the c.d.f. of a standard normal. We are interested in $P(S_{100} \geq 75)$ which is equivalent to

$$P(\frac{S_{100} - 100}{10} \geq -2.5),$$

thus the probability that the bulbs will last at least 75 years is approximately $1 - \Phi(-2.5) = \Phi(2.5)$.