

Math 201

Midterm 1 ANSWERS

November 6, 2015

1. (12 points)

A bookshelf contains 4 math books, 6 music books and 7 history books. The books are arranged in a random order and all arrangements are equally likely.

(a)(4 points) What is the total number of possible arrangements?

(b)(8 points) What is the probability that all books in each category are next to each other? Explain your answer.

Answer:

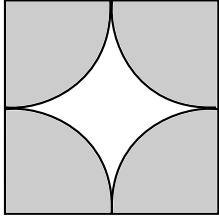
(a) If you think all the books are distinguishable, the number of possible arrangements is the same as the number of permutations of $4 + 6 + 7 = 17$ distinct objects, which is $17!$. If you think that all math books are indistinguishable, all music books are indistinguishable and all history books are indistinguishable, then you should divide by $4!6!7!$ to count for all possible ways of ordering the books in each subject. That will give $\frac{17!}{4!6!7!}$. Both answers will be accepted.

(b) First, let's count the number of ways to arrange the books so that in each category they are next to each other. There are $3!$ ways to choose the order of the categories. There are $4!$, $6!$ and $7!$ ways to arrange the math, music and history books respectively. Hence, the total number is $3!4!6!7!$. Since all outcomes are equally likely, the probability that the books in each category are next to each other is

$$\frac{3!4!6!7!}{17!}.$$

2. (11 points) A farmer is plowing a square field of side length 2 miles when he suddenly realizes he has only enough gas left to drive 1 mile. The farmer has extra gas in each corner of the field. Assuming that the current location of the farmer is uniformly distributed in the field, what is the probability that he has enough gas left to reach a corner?

Answer:



The farmer will be able to reach a corners if he is within 1 mile of a corner. That means he must be in one of the four quarter-circular regions in the figure below: .

Since the location of the farmer is distributed uniformly in the square, the probability that he is in the marked region is the total area of that region divided by the area of the square. The four quarter disks make a full disk of radius 1, hence their total area is $\pi 1^2 = \pi$. The area of the square is $2^2 = 4$. Thus, the probability the farmer gets to a corner is $\frac{\pi}{4}$.

3. (11 points) Assume the events A and B are independent. Prove that A^c and B are independent.

Answer:

By the definition of independence we need to show that

$$P(A^c)P(B) = P(A^c \cap B).$$

We can do that as follows:

$$P(A^c)P(B) = (1 - P(A))P(B) = P(B) - P(A)P(B) = P(B) - P(A \cap B) = P(A^c \cap B).$$

The independence of A and B is used for the third equality.

4. (11 points) Let X be a random variable with probability mass function

$$p_X(k) = \begin{cases} \frac{1}{9} & \text{if } k = -4, -3, -2, -1, 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}.$$

Find the probability mass function of the random variable $Y = X^2$.

Answer:

- $P(Y = 16) = P(X = 4 \text{ or } X = -4) = 2/9$
- $P(Y = 9) = P(X = 3 \text{ or } X = -3) = 2/9$

- $P(Y = 4) = P(X = 2 \text{ or } X = -2) = 2/9$
- $P(Y = 1) = P(X = 1 \text{ or } X = -1) = 2/9$
- $P(Y = 0) = P(X = 0) = 1/9$

$$p_Y(k) = \begin{cases} \frac{2}{9} & \text{if } k = 1, 4, 9, 16 \\ \frac{1}{9} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} .$$

5. (11 points) You have 3 identical-looking urns. Urn 1 contains 3 red balls. Urn 2 contains 3 green balls. Urn 3 contains 3 blue balls. Pick an urn uniformly at random and add a green ball to it. Then draw a ball from that same urn. If you draw a green ball, what is the probability the other balls in the urn are green?

Answer:

$$\begin{aligned} P(\text{other balls green} \mid \text{green}) &= P(\text{urn 2} \mid \text{green}) = \frac{P(\text{green} \mid \text{urn 2})P(\text{urn 2})}{P(\text{green})} \\ &= \frac{P(\text{green} \mid \text{urn 2})P(\text{urn 2})}{P(\text{green} \mid \text{urn 1})P(\text{urn 1}) + P(\text{green} \mid \text{urn 2})P(\text{urn 2}) + P(\text{green} \mid \text{urn 3})P(\text{urn 3})} \\ &= \frac{(1)(1/3)}{(1/4)(1/3) + (1)(1/3) + (1/4)(1/3)} = 2/3. \end{aligned}$$

6. (11 points) There are 10 balls in the basket: 5 of them are blue, 3 are red and 2 are green. 5 balls are drawn at random without replacement. What is the probability that of those five drawn there are two blue, two red and one green?

Answer:

Let Ω be the set of all subsets of 5 out of the 10 balls. Let A be the event that we draw two blue, two red and one green, then

$$P(A) = \frac{\binom{5}{2} \binom{3}{2} \binom{2}{1}}{\binom{10}{5}}.$$

Remark: This is a generalisation of the Hypergeometric distribution.

7. (11 points)

Let X and Y be independent random variables such that $X \sim \text{Geom}(1/3)$ and $Y \sim \text{Geom}(1/4)$. Define the random variable Z to be the minimum of X and Y , that is $Z = \min\{X, Y\}$. What is the probability that Z is larger than 4?

Answer:

Note that Z counts the number of trials till the first success in the series of Bernoulli independent experiments counted by X or in the one counted by Y . Since X and Y are independent we get that $Z \sim \text{Geom}(p)$, where $p = 1 - (2/3)(3/4) = 1/2$. We get that

$$P(Z > 4) = \frac{1}{2^4} = \frac{1}{16}.$$

Alternative solution: Saying that $Z > 4$ is the same as saying $X > 4$ and $Y > 4$. Since X and Y are independent, we have

$$P(Z > 4) = P(X > 4 \ \& \ Y > 4) = P(X > 4)P(Y > 4) = (2/3)^4(3/4)^4 = \frac{1}{2^4} = \frac{1}{16}.$$

8. (11 points) Which one of the following functions cannot be a cumulative distribution function? Explain your answer.

$$\text{(a)} \quad F(x) = \begin{cases} 1 - e^{-3x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

$$\text{(b)} \quad F(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{10x-x^2}{25}, & \text{if } 0 \leq x < 10, \\ 1, & \text{if } x \geq 10. \end{cases}$$

$$\text{(c)} \quad F(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{3}, & \text{if } 0 \leq x < 2, \\ \frac{1}{2}, & \text{if } 2 \leq x < 4, \\ 1, & \text{if } x \geq 4. \end{cases}$$

Answer:

The answer is (b), $\frac{10x-x^2}{25}$ is not monotone increasing for $0 \leq x < 10$.

9. (11 points) Bob, Mary and Jane's mother makes 6 sandwiches for them and give 2 sandwiches to each child. Suppose there are 2 sandwiches with grape jam, 2 with apricot jam and 2 with peach jam. What is the probability that no child got 2 sandwiches of the same kind, if all ways of distributing the sandwiches are equally likely?

Answer:

Let D be the event that no child got 2 sandwiches of the same kind and let A, B, C be the events that Bob, Mary and Jane get 2 sandwiches of the same kind respectively. We have

$$P(D) = 1 - P(A \cup B \cup C).$$

We can find $P(A \cup B \cup C)$ by the inclusion-exclusion principle. We have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC).$$

By symmetry we have that

$$P(A) = P(B) = P(C)$$

and

$$P(AB) = P(AC) = P(BC).$$

Moreover, if A and B hold then so must C , so we have

$$P(AB) = P(ABC).$$

Thus, we can write

$$P(D) = 1 - (3P(A) - 3P(AB) + P(ABC)) = 1 - 3P(A) + 2P(ABC).$$

The number of ways of distributing the sandwiches is $\frac{6!}{2!2!2!}$. The number of ways to do it so that Bob gets 2 sandwiches of the same type can be computed as follows: there are 3 ways to choose the type of sandwiches Bob gets, and once that choice is made, there are $\frac{4!}{2!2!}$ ways to give the remaining sandwiches to Mary and Jane. Thus,

$$P(A) = \frac{3 \frac{4!}{2!2!}}{\frac{6!}{2!2!2!}} = \frac{4!2!3}{6!}.$$

The number of ways to distribute the sandwiches so that each child has only one kind, i.e. so that ABC occurs, is $3!$, since the only choice we have is which type of sandwich should go to which child. Thus

$$P(ABC) = \frac{3!}{\frac{6!}{2!2!2!}} = \frac{3!(2!)^3}{6!}.$$

Combining the results, we have

$$P(D) = 1 - 3\frac{4!2!3}{6!} + 2\frac{3!(2!)^3}{6!}.$$